Research Statement

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My research interest broadly lies in the field of Theoretical Computer Science, where one is interested in understanding how much of a certain resource (like time, space, randomness) is required to complete various computational tasks. I am particularly interested in understanding the powers of different models of computation and related questions that are studied in the sub-field of Complexity Theory. Most of my work till now has been in the algebraic setting, where either the computational model or the computational task being studied is algebraic in nature.

1 Lower Bounds against Algebraic Models of Computation

The holy grail of complexity theory is to prove that $\mathsf{P} \neq \mathsf{NP}$, that is to show that there are efficiently verifiable boolean functions which can not be computed efficiently. On the other hand, one could also study the complexity of computing the formal polynomial capturing such a function instead of studying its behaviour over just the boolean domain. This approach is especially interesting since it is known that if $\mathsf{P} \neq \mathsf{NP}$, then there are "explicit" formal polynomials that can not be computed efficiently by "algebraic circuits" (assuming mild conditions). Therefore showing lower bounds on the size of *algebraic circuits* computing *explicit* polynomials is certainly a simpler task.

Algebraic Circuits are the most natural model for computing formal polynomials over some fixed field. They are directed acyclic graphs with leaves labelled by formal variables or field constants. The internal nodes are labelled by addition $(+)$ or multiplication $(\times)$ and they have the obvious operational semantics. Every node in the graph, therefore, naturally computes a polynomial over the underlying field and the polynomial computed at the node that has out-degree zero is considered to be polynomial computed by the circuit. When the underlying graph is a tree, the model is called an algebraic formula.

Another natural model of computation is that of algebraic branching programs (ABPs). They are an intermediate model between algebraic formulas and algebraic circuits. To within polynomial factors, algebraic formulas can be simulated by ABPs, and ABPs by algebraic circuits. Further, it is believed that each of the reverse transformations requires a super-polynomial blow-up in size.

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1.1 Unrestricted Models

The central lower bound question in Algebraic Complexity Theory is to show super-polynomial lower bounds against algebraic circuits and formulas. Baur and Strassen [Str73, BS83] showed that any circuit computing $\sum_{i=1}^{n} x_i^d$ must have size at least $\Omega(n \log d)$. Such lower bounds are not known against the analogous model of *boolean* circuits. However the aim is to prove super-polynomial lower bounds and, despite decades of work, this lower bound has not been improved.

For algebraic formulas, slightly better bounds are known due to Kalorkoti [Kal85], who gave an $\Omega(n^{3/2})$ lower bound for the n-variate determinant polynomial (that is the $\sqrt{n} \times \sqrt{n}$ determinant). In fact these ideas can be used to improve the bound to $\Omega(n^2 / \log n)$ for an explicit n-variate multilinear polynomial (eg. see [SY10]) . Note that with respect to algebraic formulas, $\Omega(nd)$ is a trivial lower bound for any n-variate polynomial in which every variable has individual degree at least d. Therefore the regime where the explicit hard polynomial is multilinear is particularly interesting. It turns out that in general, Kalorkoti's technique *can not* prove a lower bound which is asymptotically better than $(n^2 / \log n)$ for multilinear polynomials.

Lastly with respect to algebraic branching programs, Kumar [Kum19] gave a quadratic lower bound in the restricted setting of homogeneous ABPs. For general ABPs however, the result of Baur and Strassen [Str73, BS83] remained the best known lower bound prior to our work.

Our Contribution In a joint work with Mrinal Kumar (TIFR Mumbai), Adrian She (University of Toronto) and Ben Lee Volk (Reichman University), we show that any algebraic branching program computing the polynomial $\sum_{i=1}^{n} x_i^n$ has at least $\Omega(n^2)$ vertices. We also use our proof techniques to show a tight $\Omega(n^2)$ lower bound against algebraic formulas computing the elementary symmetric polynomial of degree $0.1n$ on n variables. Note that this lower bound is asymptotically better than $(n^2 / \log n)$, the strongest lower bound for multilinear polynomials that can be proved using previous known methods. The conference version [CKSV20] appeared in CCC 2020, and the extended version [CKSV22] appeared in Computational Complexity.

1.2 Set-Multilinear Models

A recurring theme in algebraic complexity theory is to first efficiently convert unrestricted models of computation to special kinds of syntactically-restricted models, show strong lower bounds against them and then recover non-trivial lower bounds against the original models using the efficiency of this conversion. One such restriction is that of set-multilinearity. A polynomial over a set of variables $\{X_i = \{x_{i,j} : j \in [n_i]\} : i \in [d]\}$ for some $d, n_1, \ldots, n_d$ is said to be set-multilinear with respect to the partition $\{X_1, \ldots, X_d\}$ if every monomial in it has exactly one variable each from $X_1, \ldots, X_d$. Syntactically set-multilinear models are those that have syntactic restrictions which ensures that they compute set-multilinear polynomials at every node.

The proof strategy mentioned earlier has been used most remarkably in the work of Limaye, Srinivasan and Tavenas [LST21], where they prove the first super-polynomial lower bound against algebraic circuits of *any* constant depth. They first show that constant-depth a algebraic circuit can be converted to a *set-multilinear* algebraic formula of constant depth without blowing up the size much (as long as the degree of the polynomial computed is small) and then they show strong lower bounds against low-depth set-multilinear circuits (for a polynomial of small enough degree).

Very recently, Bhargav, Dwivedi and Saxena [BDS23] proved a statement similar to the first part of the proof strategy in [LST21]. They showed that an unrestricted ABP can be converted to a "sum of ordered set-multilinear ABPs" without blowing up the size by much, as long as the degree of the polynomial computed is small. Even though ordered set-multilinear ABPs are completely understood as far as lower bounds are concerned, super-polynomial lower bounds against a sum of such ABPs were known in only extremely restricted settings [AR16, BDS23].

Our Contribution In a joint work with Deepanshu Kush (University of Toronto), Shubhangi Saraf (University of Toronto) and Amir Shpilka (Tel Aviv University), we show the first lower bound against sums of ordered set-multilinear ABPs. In fact, we show that there is a polynomial that can be computed by ABPs such that any sum of ordered set-multilinear ABP computing it has exponential size. Further the lower bound stays super-polynomial as long as the degree is barely larger than the degree requirement of [BDS23] for it to translate into a general ABP lower bound. The proofs follow along the same lines as those in the works of Kush and Saraf [KS22, KS23] and the manuscript [CKSS23] is currently under submission.

1.3 Non-Commutative Models

Another natural restriction, that is studied widely, is non-commutativity of multiplication [Hya77, Nis91]. In this setting, $XY \neq YX$ for indeterminates X and Y[1]. Non-commutative models are extremely restricted syntactically and lower bounds against these models do not directly lead to lower bounds in the more general commutative setting. However, because of the similarities with the set-multilinear models, there has been a recent resurgence of interest in proving lower bounds against non-commutative models [TLS22, FLM$^+$23, FLST23].

In spite of the heavy syntactic restrictions, the best lower bound known for circuits in this setting is still the one by Baur and Strassen [Str73, BS83]. Interestingly, Carmossino, Impagliazzo, Lovett and Mihajlin [CILM18] showed that super-linear lower bounds against non-commutative circuits for constant degree polynomials would result in exponential lower bounds against the same model. Therefore, proving super-linear lower bounds for constant degree polynomials might be hard. But, for example, can we improve the lower bound of $\Omega(n \log d)$ due to Baur

[1]This is indeed the case, for example, when X, Y were replaced by matrices.

Strassen to $\Omega(nd)$? Even when the circuits are assumed to be *homogeneous*[2], such a lower bound was not known prior to our work.

On the other hand, exponential lower bounds are known against non-commutative formulas [Nis91]. However, Nisan's proof actually works for ABPs as well. As mentioned earlier, in at least the general setting, ABPs are believed to be computationally more powerful than formulas. A natural question therefore, posed by Nisan [Nis91] himself, is whether one can show a super-polynomial separation between ABPs and formulas in the non-commutative setting. In a remarkable recent result, Limaye, Srinivasan and Tavenas [TLS22] showed that there is indeed a super-polynomial separation between homogeneous formulas and ABPs in the non-commutative setting. In fact, very recently, the same authors along with Fournier [FLST23] improved this result to show that there is a polynomial that can be computed by in-homogeneous non-commutative formulas such that any homogenous non-commutative formula computing it must have super-polynomial size. The general question, however, continues to remain open.

Our Contribution We provide a new approach towards resolving Nisan's conjecture by studying certain structured formulas which we call *abecedarian* formulas. Firstly, we show a super-polynomial separation between non-commutative ABPs and abecedarian formulas. We then go on to show that for certain settings of parameters, proving lower bounds against abecedarian formulas is enough to prove lower bounds against general formulas. Abecedarian formulas are not necessarily homogeneous, and hence this work is incomparable to the work of [LST22, FLST23]. The conference version [Cha21] appeared in the proceedings of CCC 2021.

In a joint work with Pavel Hrubeš (Czech Academy of Sciences), we show a tight $\Omega(n^2)$ lower bound against homogeneous non-commutative circuits computing the ordered elementary symmetric polynomial of degree $0.5n$ on n variables. Further, this polynomial can be computed by an in-homogenous non-commutative circuit of size $O(n \log^2 n \log \log n)$. This is the first instance of a super-linear separation between homogenous and in-homogenous circuits. The conference version [CH23] appeared in the proceedings of CCC 2023.

1.4 Classes Beyond VNP

As mentioned earlier, the central questions in algebraic circuit complexity involve proving lower bounds for *explicit* polynomials, or polynomials in VNP — the algebraic analogue of NP. However, there are classes of polynomials that are possibly larger than VNP. One such class is $\mathsf{VPSPACE}$. First defined in the work of Koiran and Perifel [KP09a, KP09b], as the set of polynomials whose coefficients can be computed in $\mathsf{PSPACE}/\operatorname{poly}$, the class has various equivalent definitions which are completely algebraic [Poi08, Mal11]. Koiran and Perifel [KP09a, KP09b] showed that if $\mathsf{VNP} \neq \mathsf{VPSPACE}$, then $\mathsf{P} \neq \mathsf{PSPACE}/\operatorname{poly}$. It would, therefore, be interesting to ask if $\mathsf{VNP} \neq \mathsf{VPSPACE}$.

[2]Every node computes a homogenous polynomial.

Our Contribution In a joint work with Kshitij Gajjar (IIT Jodhpur) and Anamay Tengse (Reichman University), we study this question in the *monotone setting* — where the model of computation is only allowed to use non-negative field constants to compute the polynomial. Being an extremely restricted model of computation, very strong lower bounds are known in this setting. For instance, Yehudayoff [Yeh19] showed an exponential lower bound against monotone algebraic circuits for a polynomial contained in the monotone analogue of VNP (or mVNP).

We study the natural monotone analogues of the various algebraic definitions of VPSPACE due to Poizat [Poi08] and Malod [Mal11]. We show that, unlike the non-monotone setting, these definitions are not equivalent. Finally, because of its closure properties and the fact that the permanent polynomial can be computed by this model, we define the monotone analogue of Poizat's definition to be monotone VPSPACE (or mVPSPACE) and show that this class is strictly larger than mVNP. The conference version [CGT23] appeared in the proceedings of FSTTCS 2023.

1.5 Ongoing Work and Future Directions

My main interest lies in proving lower bounds for various algebraic models. Some of the concrete questions I am thinking about currently are listed below.

I want to continue studying algebraic formulas and ABPs both in the general and restricted settings. In fact, the recent remarkable works of Limaye, Srinivasan and Tavenas [LST21, TLS22] and the later follow up works [KS22, LST22, KS23, FLM+23, FLST23] suggest that we might be close to proving strong lower bounds against homogeneous formulas. With respect to algebraic branching programs, a natural question is to prove a super quadratic lower bound (even in the homogeneous case). I believe a natural first step towards this would be to prove such a bound against homogeneous multilinear ABPs.

In the non-commutative setting, Nisan's question [Nis91] of whether ABPs can be efficiently simulated by algebraic formulas, continues to intrigue me. I am also interested in proving super-polynomial lower bounds against non-commutative circuits by working towards closing the gap between our result and the hardness escalation result in [CILM18]. I am working with Abhranil Chatterjee (ISI Kolkata) on both these problems.

2 Other Projects

2.1 Algebraically Natural Proofs

As we just saw, progress on the central question of proving lower bounds against general algebraic models has been painfully slow. On the other hand, in their remarkable recent paper, Limaye, Srinivasan and Tavenas [LST21] showed a super-polynomial lower bound against constant depth circuits. This has led to some work towards understanding whether certain *natural* techniques that have helped in proving lower bounds for restricted models can be useful in proving lower bounds against general models.

Analogous to the work of Razborov and Rudich [RR97] in the boolean setting, Forbes, Shpilka and Volk [FSV18], and Grochow, Kumar, Saks and Saraf [GKSS17], proposed the framework of algebraically natural proofs. They showed that almost all of the known strategies for proving lower bounds against restricted models implicitly go via defining a property for the set of all polynomials, which is then used to separate the hard polynomial from the easy ones. The property is usually captured via *zeroes of efficiently computable polynomial equations*. Forbes *et al.* [FSV18] also gave weak evidence *against* the existence of natural proofs for proving super-polynomial lower bounds against general circuits. However, the question of whether they exist or not remains open.

Our Contribution In a joint work with Mrinal Kumar (TIFR Mumbai), C. Ramya (IMSc), Ramprasad Saptharishi (TIFR Mumbai) and Anamay Tengse (Reichman University), we provide some evidence that algebraically natural proofs for proving lower bounds against polynomial sized circuits *exist*. We do so by showing that if the target hard polynomial came from a sufficiently rich restricted class, then there exist efficiently computable polynomial equations that can distinguish between it and polynomials that can be computed by polynomial sized circuits. In particular this restricted class contains the permanent polynomial, which is believed to require super-polynomial sized circuits [Val79]. The conference version [CKR+20] appeared in FOCS 2020.

There has also been some follow-up work by the other authors [KRST22]. They show that *assuming the permanent requires exponential sized circuits, if* VP *has natural proofs, then there is also a natural proof that separates* VP *and* VNP. Very recently, in a follow-up work with Anamay Tengse (Reichman University), we show that proving an algebraic natural proofs barrier would imply either $\mathsf{VP} \neq \mathsf{VNP}$ or $\mathsf{DSPACE}(\log^{\log^* n} n) \not\subset \mathsf{P}$. The manuscript [CT23] is under submission.

2.2 Algebraic Independence Testing and Constructing Faithful Homomorphisms

We now look at a more algorithmic question. A set of polynomials $\{f_1, \ldots, f_m\}$ is said to be *algebraically dependent* if there is some non-zero polynomial combination of them that is zero. For example, if $f_1 = x$, $f_2 = y$ and $f_3 = x^2 + y^2$, then $A = z_1^2 + z_2^2 - z_3$ is an *annihilator*. We note that the underlying field is very important in this case. For example, $x + y$ and $x^p + y^p$ are algebraically independent when considered as polynomials over $\mathbb{R}$ or $\mathbb{C}$, but they are algebraically dependent over $\mathbb{F}_p$ since $(x+y)^p = x^p + y^p$. The obvious question at this point is thus the following: *Is there an* efficient *algorithm to check whether a given set of polynomials are algebraically independent*?

Surprisingly, this is not known in general. Over fields of characteristic zero, a classical result of Jacobi [Jac41] leads to a randomized polynomial time algorithm for algebraic independence testing. However, the algorithm fails over fields of finite characteristic.

Over such fields, Pandey, Saxena and Sinhababu [PSS18] characterised the extent of failure of the Jacobian criterion using the notion of *inseparable degree* of the given set of polynomials. They presented a randomised algorithm to solve the problem in general. Unfortunately, the algorithm is efficient only when the inseparable degree is bounded by a constant. Later, Guo, Saxena and Sinhababu [GSS19] showed that the problem of algebraic independence testing in $\mathsf{AM} \cap \mathsf{co\text{-}AM}$ via a very different approach that does not depend on the inseparable degree.

Application to Polynomial Identity Testing Given an algebraic circuit as input, the Polynomial Identity Testing (PIT) problem asks whether the polynomial computed by it is identically zero. It is well known that this problem has an efficient randomised polynomial-time algorithm. A central algorithmic question in Algebraic Complexity Theory is whether this can be derandomised.

Beecken, Mittman and Saxena [BMS13] showed that the concept of algebraic independence has a connection with the question of PIT via the notion of *faithful homomorphisms* (or faithful maps). They used the results in [Jac41] to construct such maps over fields of zero characteristics, which were then used by them and Agrawal, Saha, Saptharishi and Saxena [ASSS16] to design identity tests for several classes of algebraic circuits. These results however do not carry over to fields of finite characteristic, since the Jacobian criterion fails over such fields.

Our Contribution In a joint work with Ramprasad Saptharishi (TIFR), we addressed the question of constructing faithful maps over finite characteristic in several restricted cases. Using the characterisation given by Pandey *et al.*[PSS18], we provide a recipe for constructing faithful maps for polynomials coming from certain circuit classes (such as sum of sparse polynomials etc.). The techniques used are similar to those in [BMS13] and [ASSS16], and lead to polynomial identity tests for these classes. The conference version [CS19] appeared in the proceedings of FSTTCS 2019, and the extended version [CS23] appeared in the ACM Transactions on Computation Theory.

2.3 Shortest Paths with Dynamic Edge Weights

Suppose that there is a road network with n traffic signals such that the amount of traffic on each road varies as a linear function of time, called the parameter. Then the shortest path from a source s to a destination t might be different at different points of time.

Gusfield [Gus80] showed that the number of different shortest paths between s and t cannot be more than $n^{O(\log n)}$. This problem was further studied by Carstensen [Car83] and later by Mulmuley and Shah [MS01]. They both exhibited road networks for which the number of different paths is $n^{\Omega(\log n)}$, showing that Gusfield's upper bound is tight (up to a constant factor in the exponent).

The assumption of the edge weights being univariate linear functions, however, is very simplistic for practical purposes. A natural question is what happens if the edge weights are allowed to be multivariate polynomials. Recently, Gajjar and Radhakrishnan [GR19] showed that if the edge weights are linear forms in up to three variables instead, then the number of different shortest paths is at most $n^{O(\log^2 n)}$. Barth, Funke and Proissl [BFP22] extend this result to give an $n^{O(\log^{m-1} n)}$ upper bound when the edges weights are linear forms in m varibales.

Another natural offshoot of the shortest path problem is that of *time dependent shortest paths* (TDSPs). The problem has been well studied for univariate monotone linear edge weights [DOS12, FHS14]. We study the case when the edge weights are non-monotone and also when they are non-linear. Further, we also define the notion of *generalised parametric path* (GPP) and work with weight functions in higher dimensions. That is the weight on an edge e looks like $\{w_e : \mathbb{R}^k \to \mathbb{R}^k\}$. Apart from generalising TDSPs, they also capture other naturally occurring problems like financial networks and arbitrage networks [HP14, Hau14, Moo03].

Our Contribution With respect to the GPP problem, we extend the known results to show even in the non-monotone linear version, there is an efficient algorithm to output the shortest path. We also show that in this setting, as the parameter varies, there are at most quasi-polynomially (in terms of the number of vertices) many distinct shortest paths possible for any graph. However, when the edge weights are allowed to vary even quadratically, we show that it is NP-complete to even approximate the shortest path to within a constant factor. We also show that in this setting there are n-vertex graphs in which as the parameter value varies, there are $\exp(\Omega(n))$ different shortest paths. Finally, when the edge weigts are allowed to be in higher dimension, we show that it is NP-hard to find the shortest path. This is joint work with Kshitij Gajjar (IIT Jodhpur), Jaikumar Radhakrishnan (ICTS-TIFR) and Girish Varma (IIIT Hyderabad). The conference version [GVCR21] appeared in the proceedings of UAI, 2021.

With respect to the parametric shortest path problem with edge weights allowed to be linear polynomials over larger number of variables, we reprove the statement of [BFP22]. Apart from being able to achieve marginally better parameters, we believe our proof is more intuitive. This is an ongoing joint work with Kshitij Gajjar (IIT Jodhpur), Jaikumar Radhakrishnan (ICTS-TIFR).

2.4 Future Directions

Apart from my core interest in proving lower bounds against models of algebraic computation, I enjoy working on projects from different areas in theoretical computer science. Most questions that are complexity theoretic in nature interest me and I am open to working on any interesting problem irrespective of the specific area it belongs to.

One such specific model I am interested in currently is *hazard-free computation*. Informally, if the output of a certain boolean function is not sensitive to the value of certain input bits, then a circuit computing the function is hazard-free if it continues to output the correct answer even if any of those input bits is set to "undefined". I am currently trying to understand the recent works on hazard-free circuits [IKL$^+$19, KS20, IKS23] to find interesting related questions.

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