A Quadratic Lower Bound against Algebraic Branching Programs

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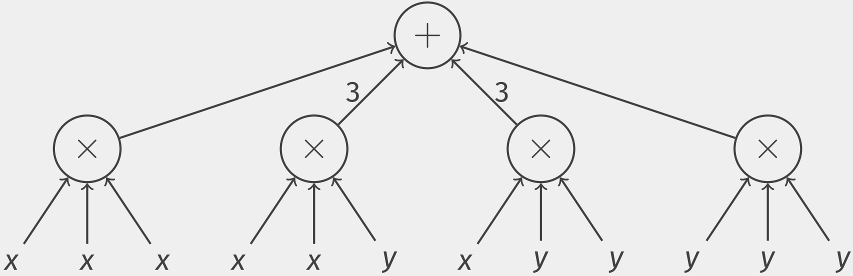

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Algebraic Formulas and Algebraic Circuits

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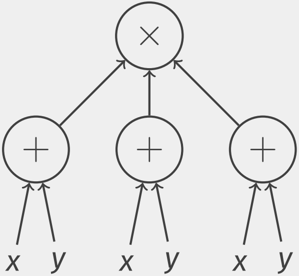

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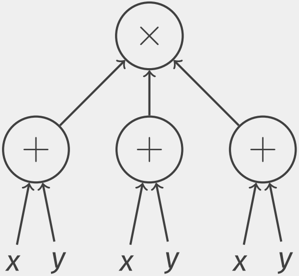

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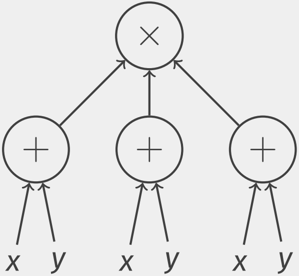

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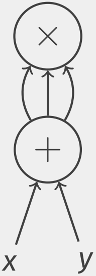

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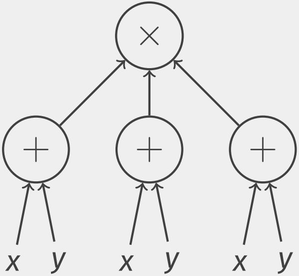

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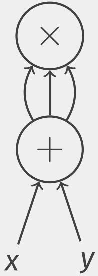
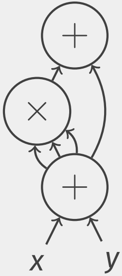


$(x+y)^3+(x+y)$



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VF $\subseteq$ VP

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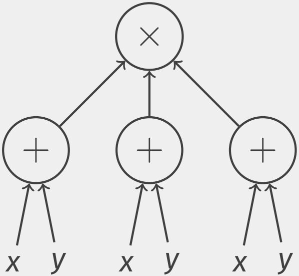

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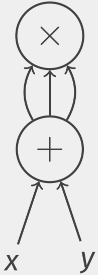
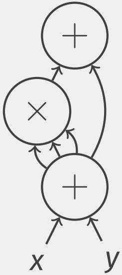
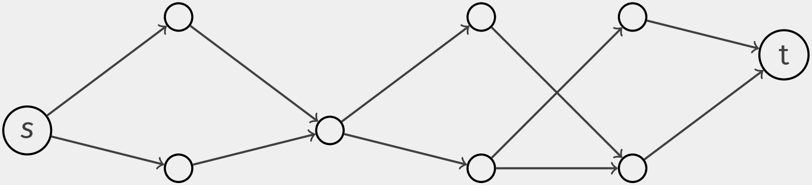
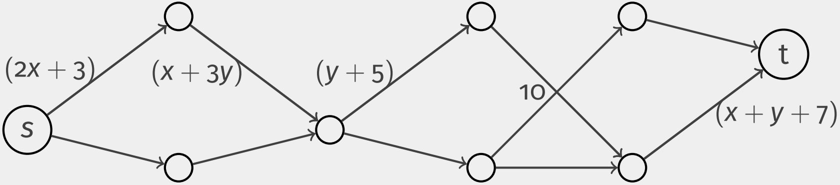


$(x + y)^3 + (x + y)$



Algebraic Branching Programs





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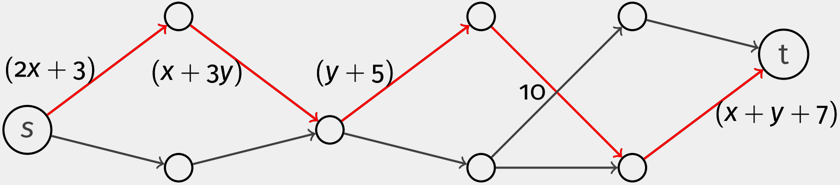

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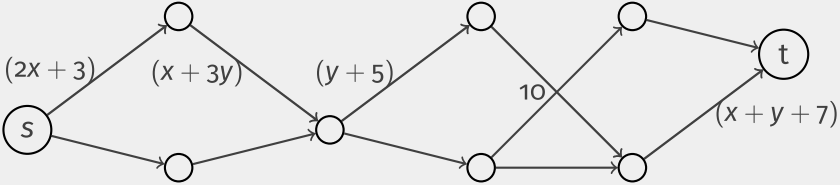

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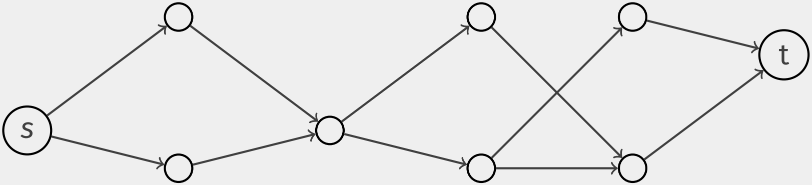

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where $A_i(0) = 0 = B_i(0)$ and $\deg(\delta(\mathbf{x})) < d$, has at least

$$((n/2) - r) \cdot (d-1) \quad \text{vertices.}$$

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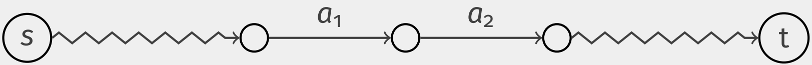
no. of error terms = $r_{\ell+1} \leq r_\ell + \frac{s_\ell}{d_\ell/3}$

Proof of the Induction Step



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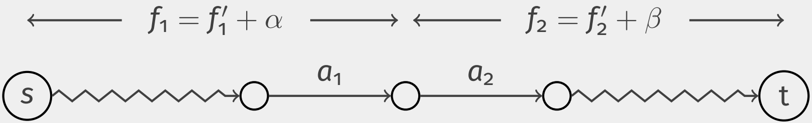
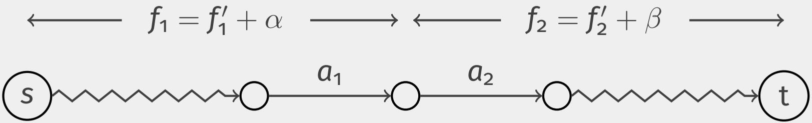


$\mathcal{A}_\ell = f_1 \cdot f_2$



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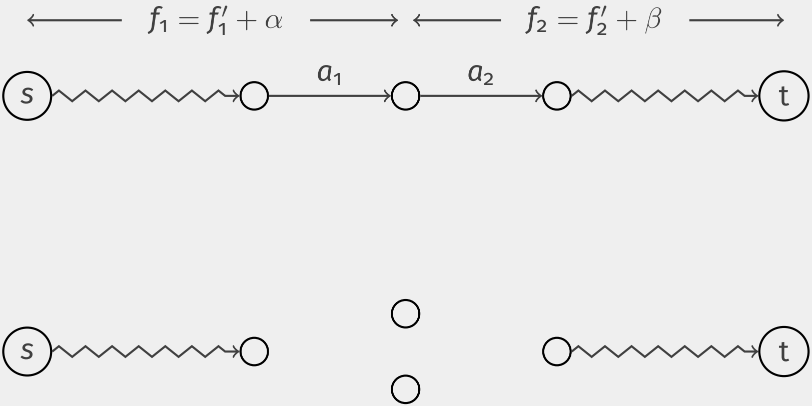

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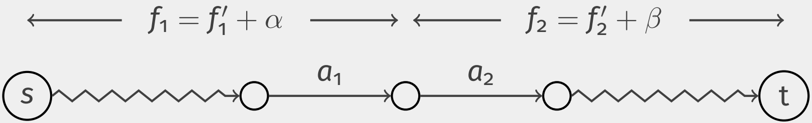

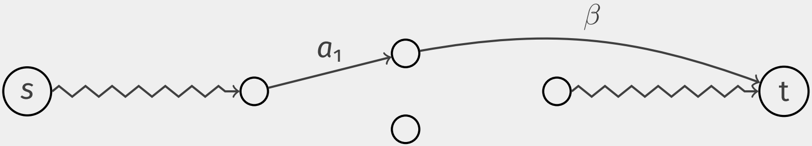

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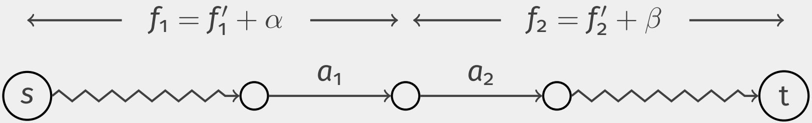

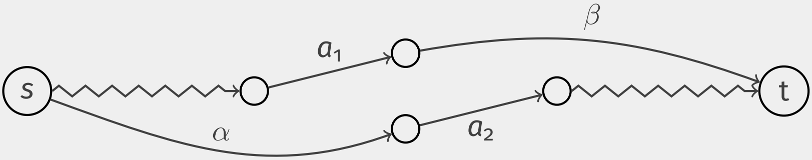

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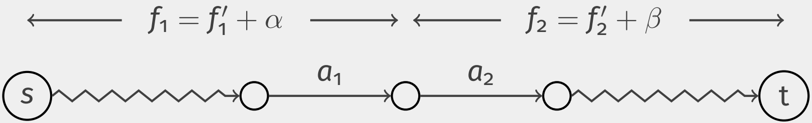

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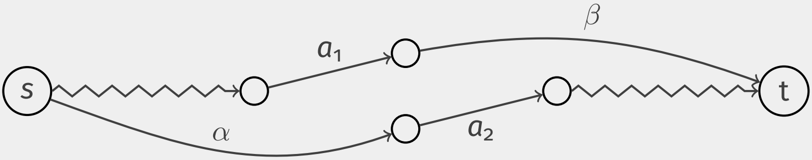

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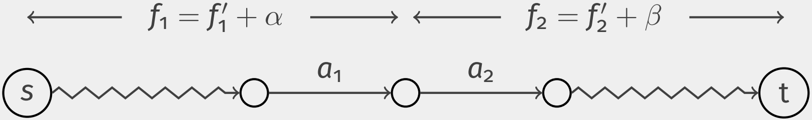

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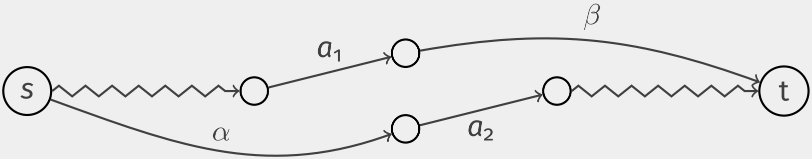

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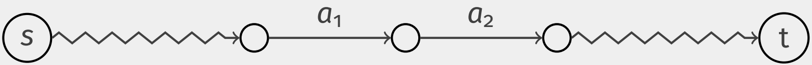

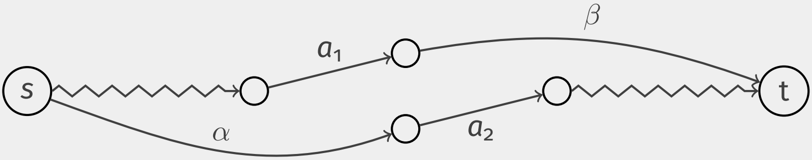

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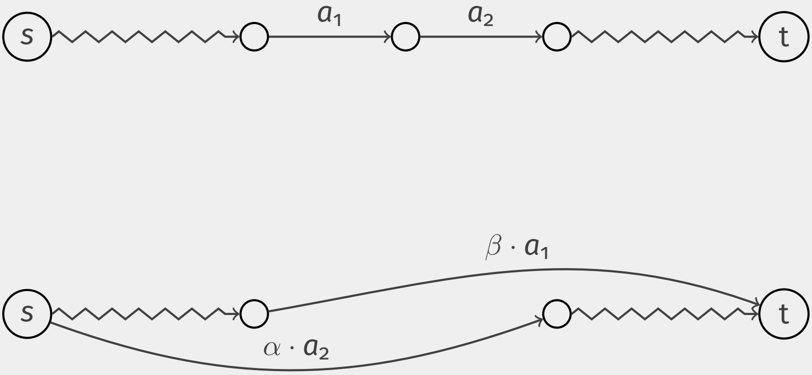

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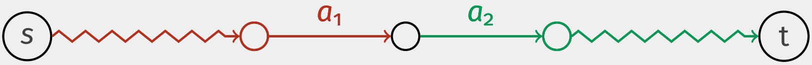

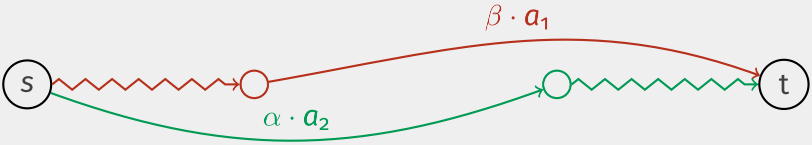

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2. For unlayered ABPs with edge labels of degree $\leq \Delta$, the lower bound we get is $\Omega(n \log n / \Delta \log \log n)$.
3. The lower bound is also true for a multilinear polynomial

$$\mathrm{ESym}(n, 0.1n) = \sum_{i_1 < \cdots < i_{0.1n} \in [n]} \prod_{j=1}^{0.1n} x_{i_j}.$$

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- Multilinearising the SY polynomial gives $\Omega(n^2/\log n)$.
- Kalorkoti's method can not give better than $\Omega(n^2/\log n)$.

Our Result: Any formula computing $\mathrm{ESym}_{n,0.1n}$ has $\Omega(n^2)$ vertices, where

$$\mathrm{ESym}(n, 0.1n) = \sum_{i_1<\cdots<i_{0.1n}\in[n]} \prod_{j=1}^{0.1n} x_{i_j}.$$

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Thank you!