A Quadratic Lower Bound against Algebraic Branching Programs

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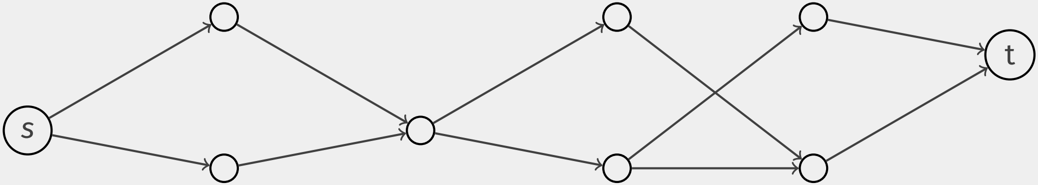
Algebraic Branching Programs



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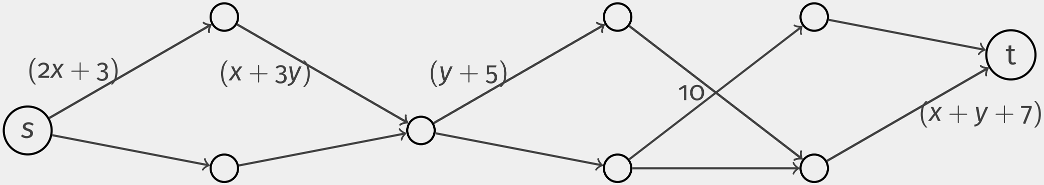

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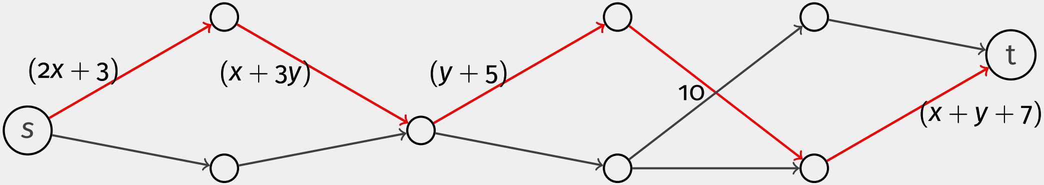

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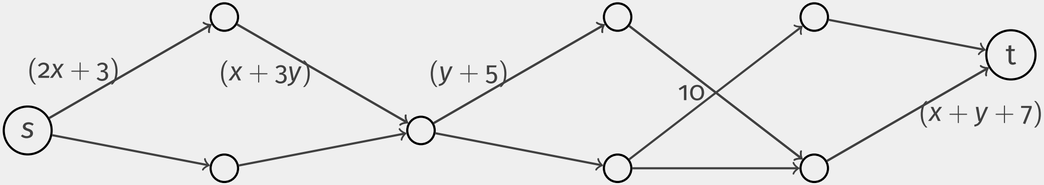

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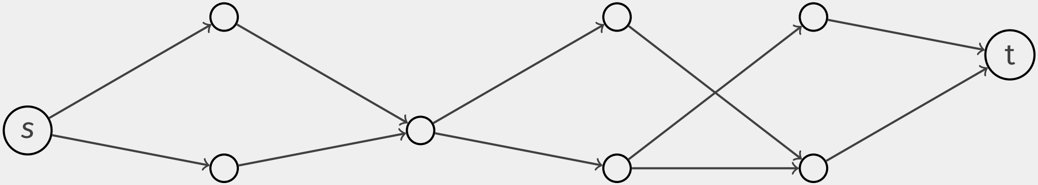

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Central Question: Can one give an n-variate, degree d polynomial that can not be represented by an ABP of size $\mathrm{poly}(n, d)$?

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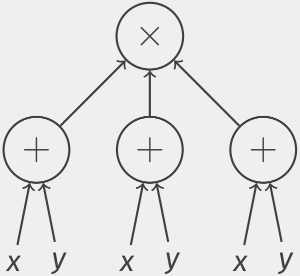

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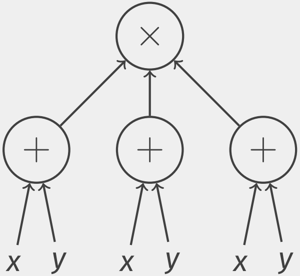

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Algebraic Formulas and a Lower Bound against them

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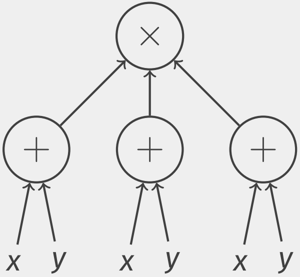

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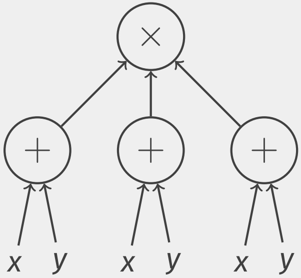

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Our Result: Any formula computing $\text{ESym}_{n,0.1n}$ requires $\Omega(n^2)$ vertices, where

$$\text{ESym}(n, 0.1n) = \sum_{i_1<\cdots<i_{0.1n}\in[n]} \prod_{j=1}^{0.1n} x_{i_j}.$$

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3. The lower bound is also true for the elementary symmetric polynomial.

$$\text{ESym}(n, 0.1n) = \sum_{i_1 < \cdots < i_{0.1n} \in [n]} \prod_{j=1}^{0.1n} x_{i_j}$$

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Thank you!