Separating ABPs and some Structured Formulas in the Non-Commutative Setting

Prerona Chatterjee

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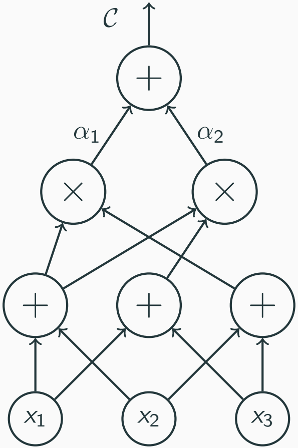
July 20, 2021

Algebraic Circuits and Formulas



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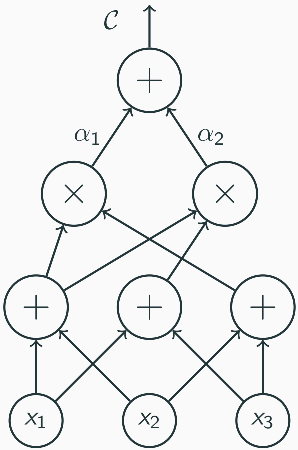

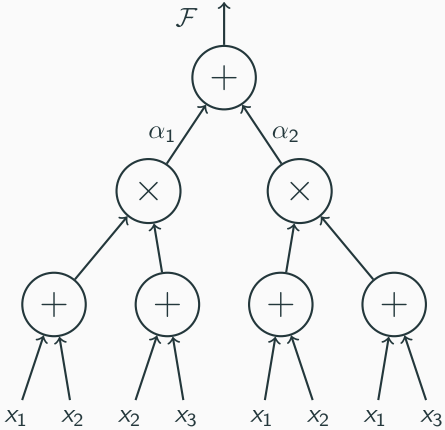

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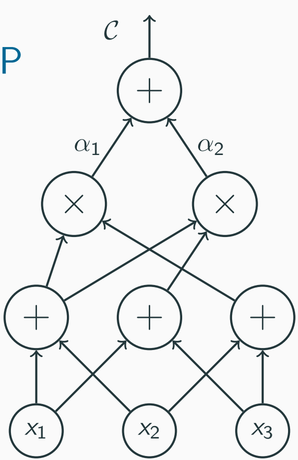
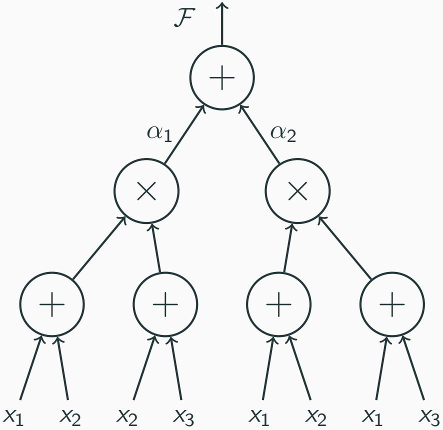
VP





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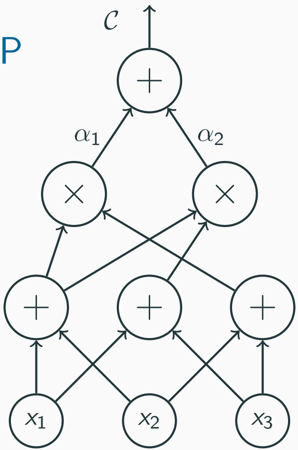

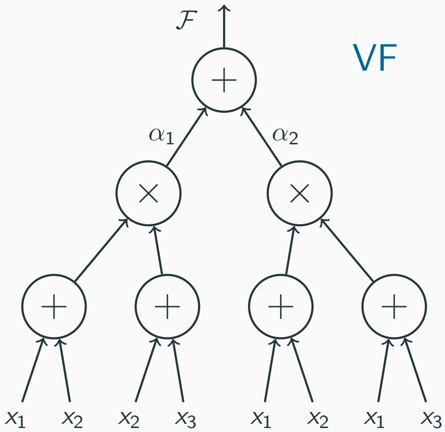

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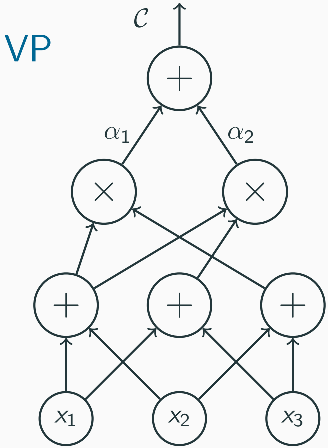
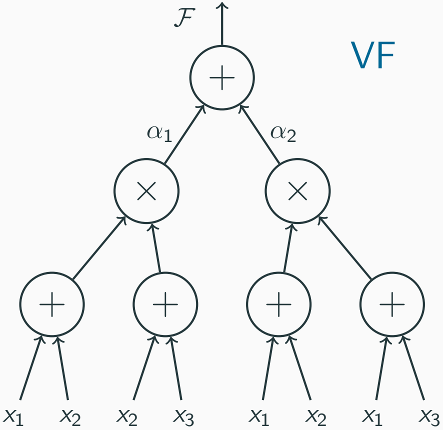


$\boxed{\mathrm{VP} \supseteq \mathrm{VF}}$



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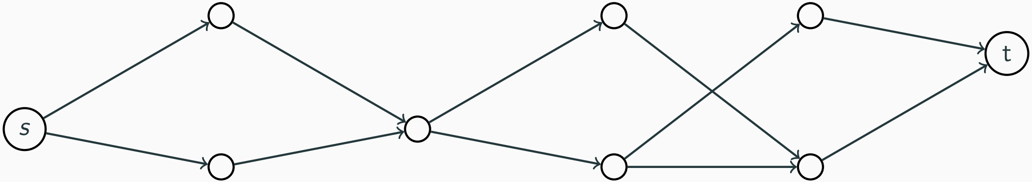
$$\mathsf{VF}_{\mathsf{nc}} \subseteq \mathsf{VBP}_{\mathsf{nc}} \subseteq \mathsf{VP}_{\mathsf{nc}}$$

So Nisan actually showed that $\mathsf{VBP}_{\mathsf{nc}} \neq \mathsf{VP}_{\mathsf{nc}}$.

Algebraic Branching Programs



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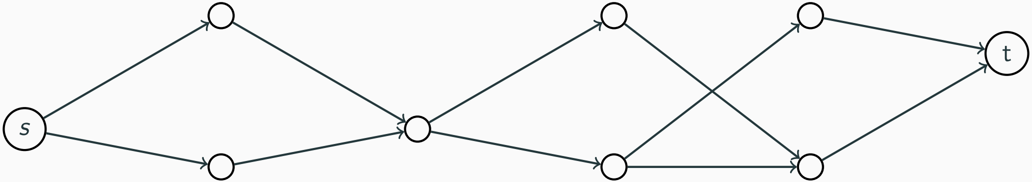

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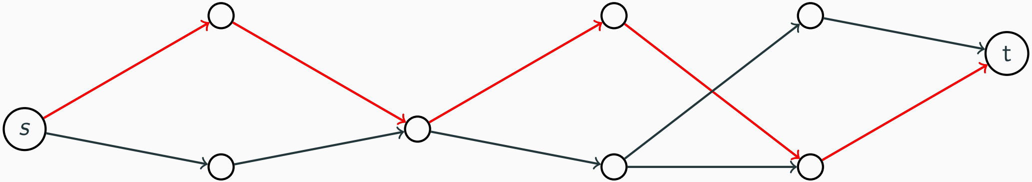

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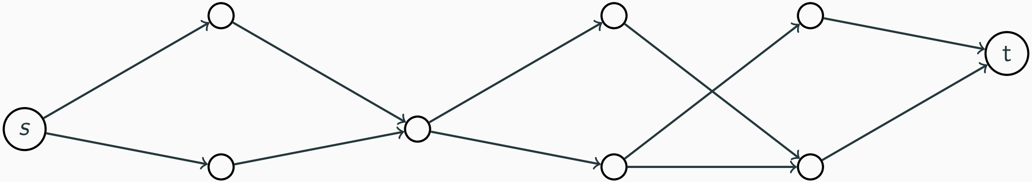

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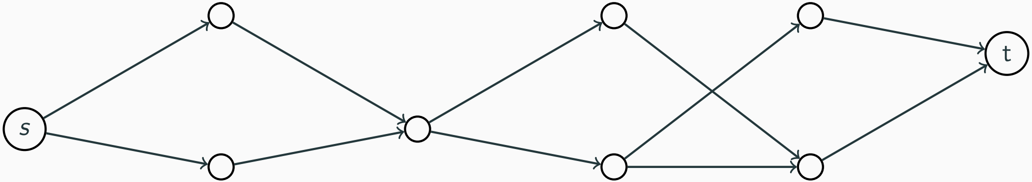

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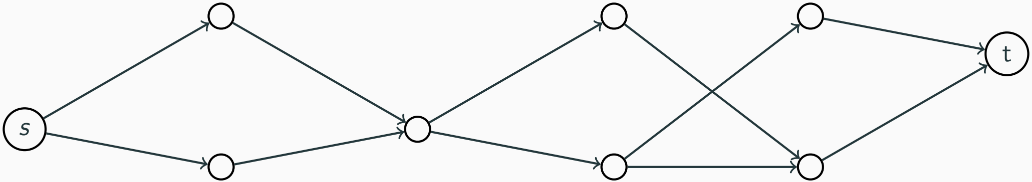

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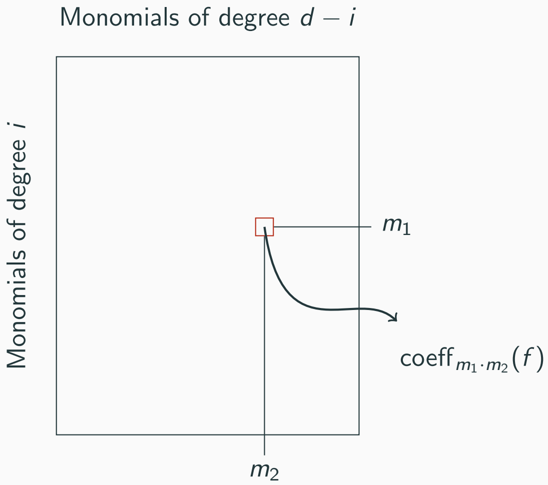
For a general polynomial f of degree d, $f = \mathrm{Hom}_0(f) + \mathrm{Hom}_1(f) + \cdots + \mathrm{Hom}_d(f)$.

Nisan's Characterisation



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For every $1 \leq i \leq d$, consider the matrix $M_f(i)$ described alongside.

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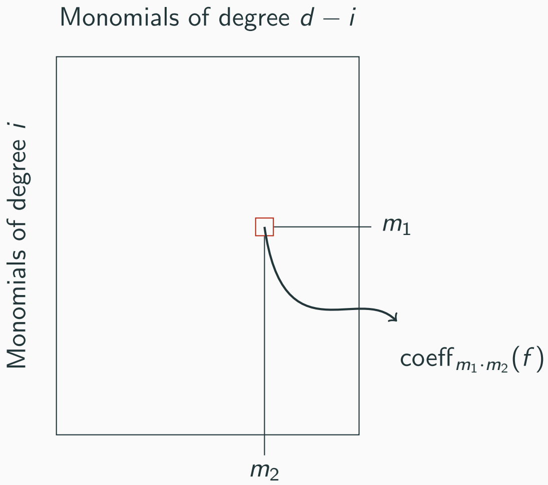

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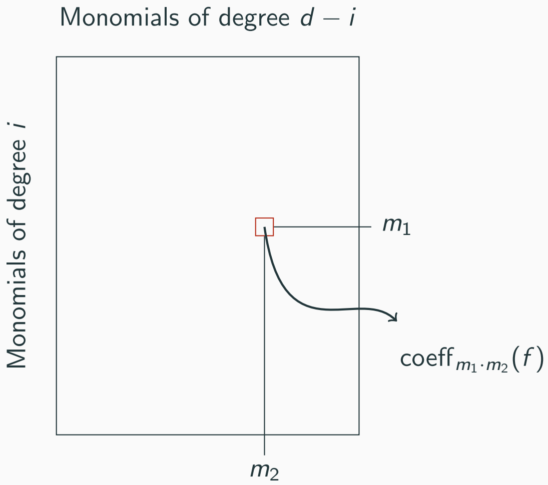

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If $\mathcal{A}$ is the smallest ABP computing f,

$$\mathrm{size}(\mathcal{A}) = \sum_{i=1}^{d} \mathrm{rank}(M_f(i)).$$

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Main Result:

There is a tight superpolynomial separation between *abecedarian* formulas and ABPs.

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Generalises the notion of ordered polynomials (defined in [HWY11]).

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$$\mathrm{Det}_n(\mathrm{x}) = \sum_{\sigma \in S_n} (-1)^{\mathrm{sgn}(\sigma)} x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$$

Buckets	Example
$\{X_i\}_{i\in[n]}$ where $X_i = \{x_{ij}\}_{j\in[n]}$	$\mathrm{Det}_n(\mathrm{x})$

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$$f(\mathrm{x}) \quad \xrightarrow[\text{in ascending order}]{\text{Order the monomials}} \quad f^{(nc)}(\mathrm{x})$$

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Note:

$$\mathrm{ESYM}_{n,d}^{(\mathrm{ord})} = \sum_{1\le i_1<\ldots<i_d\le n} x_{i_1}^{(1)} \cdots x_{i_d}^{(d)}$$

is abecedarian w.r.t. both $\left\{X_k = \left\{x_i^{(k)}\right\}_{i\in[n]}\right\}_{k\in[d]}$ as well as $\left\{X_i = \left\{x_i^{(k)}\right\}_{k\in[d]}\right\}_{i\in[n]}$.

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1. There is an explicit n^2-variate, degree d polynomial $f_{n,d}(\mathsf{x})$ which is abecedarian with respect to a partition of size n such that

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2. Let f be an n-variate abecedarian polynomial with respect to a partition of size $O(\log n)$ that can be computed by an ABP of size $\mathrm{poly}(n)$. A super-polynomial lower bound against abecedarian formulas for f would imply that $\mathrm{VF}_{\mathrm{nc}} \neq \mathrm{VBP}_{\mathrm{nc}}$.

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A positive answer to either of these questions would imply that $\mathrm{VBP}_{\mathrm{nc}} \neq \mathrm{VF}_{\mathrm{nc}}$.

The Explicit Statement for the Separation

$$\text{linked\_CHSYM}_{n,d}(x) = \sum_{i_0=1}^{n} \left(\sum_{i_0 \leq i_1 \leq \ldots \leq i_d \leq n} x_{i_0,i_1} \cdot x_{i_1,i_2} \cdots x_{i_{d-1},i_d} \right)$$

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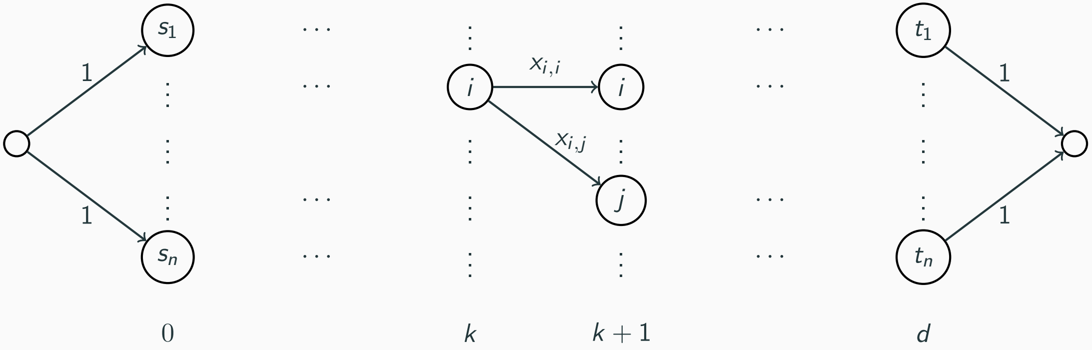

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If there is a homogeneous structured abecedarian formula of size s computing $h_{n/2,d}(\mathsf{x})$ and a homogeneous abecedarian formula of size s' computing $\mathrm{CHSYM}_{n/2,d'}(\mathsf{x})$, then there is a homogeneous abecedarian formula computing $\mathrm{CHSYM}_{n/2,d\cdot d'}(\mathsf{x})$ of size $s \cdot s'$.

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Note: The last step uses ideas similar to those used by Raz to *multilinearise* formulas. This is why the transformation is efficient only when the number of buckets in the partition is small.

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Thank you!