Separating ABPs and some Structured Formulas in the Non-Commutative Setting

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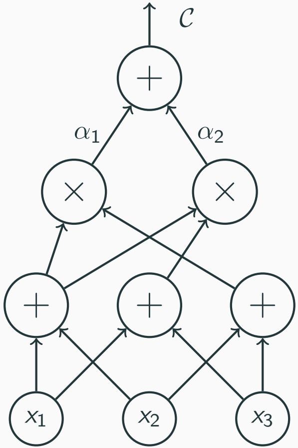
Algebraic Circuits and Formulas



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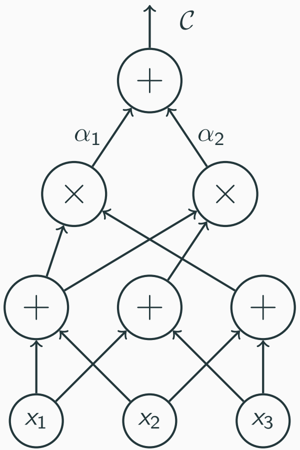

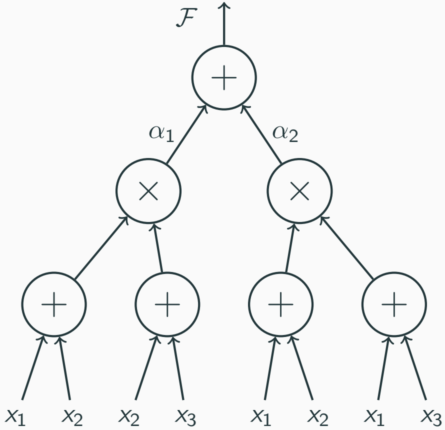

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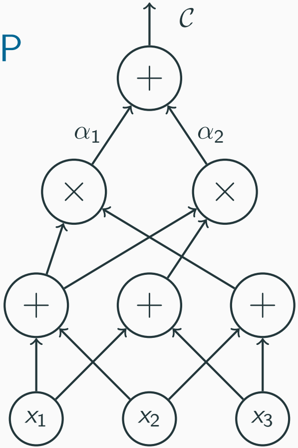
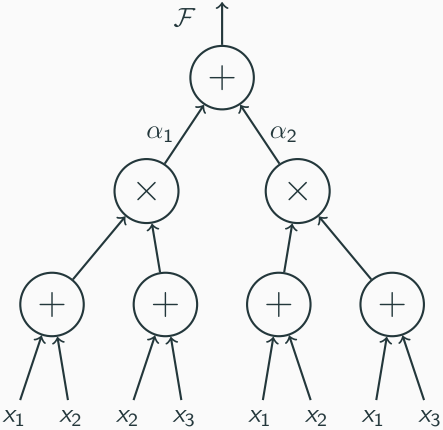
VP





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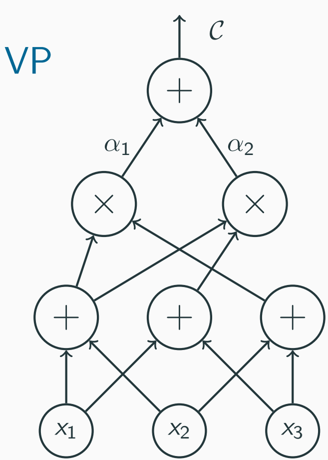

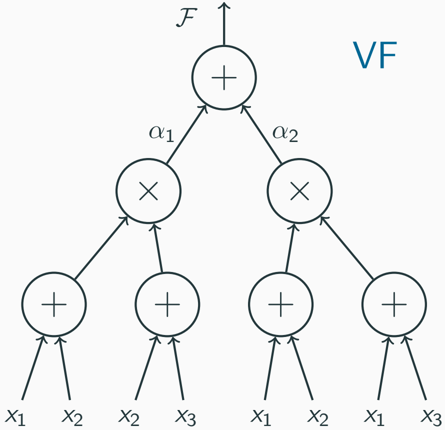

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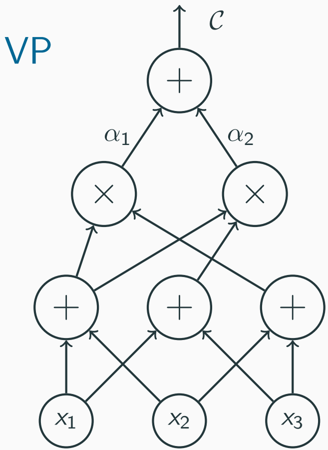
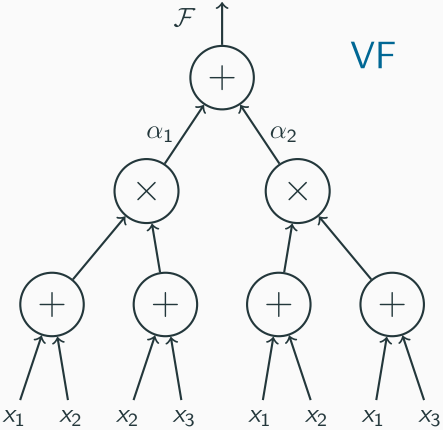


VP ⊇ VF



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$$f(x, y) = (x + y)^2 = (x + y) \times (x + y) = x^2 + xy + yx + y^2 \neq x^2 + 2xy + y^2$$

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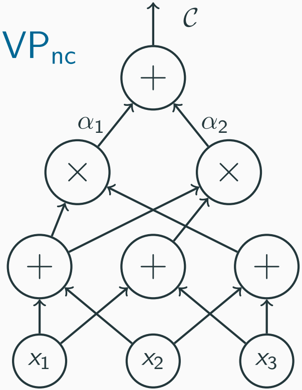

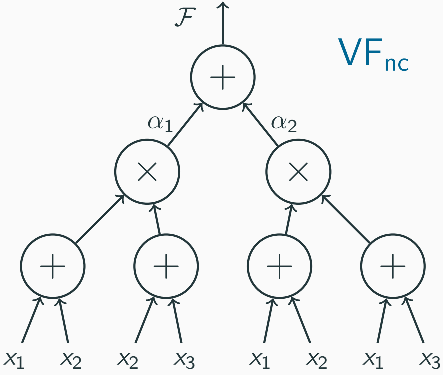

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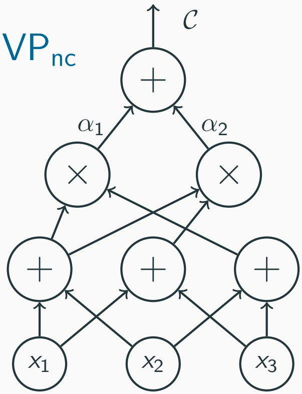
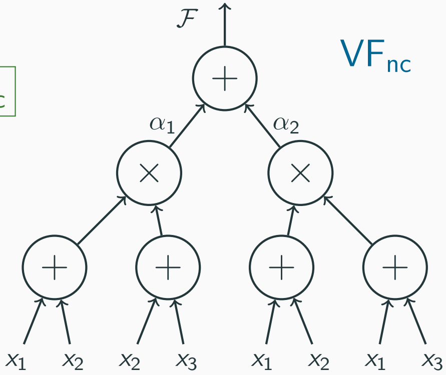


$\mathsf{VP}_{\mathsf{nc}} \supseteq \mathsf{VF}_{\mathsf{nc}}$



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Main Result:

There is a tight superpolynomial separation between *abecedarian* formulas and ABPs.

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3. Convert the structured abcd-formula into a homogeneous multilinear formula.
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Concluding Remarks

Studying abcd-formulas might be enough: For any n-variate polynomial that is abecedarian with respect to a partition of size $O(\log n)$, if it can be computed efficiently by formulas, then they can also be computed efficiently by abcd-formulas.

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Super-polynomial separation between homogeneous formulas and ABPs [LST21]: There is an n^2-variate degree-n polynomial that can be computed by a homogeneous ABP of size $\Theta(n^2)$ but any homogeneous formula computing it must have size $n^{\Omega(\log \log n)}$.

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Question: Can ideas from [LST21] and this paper be combined to answer Nisan's question?

Thank You !

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