Hardness and Independence of Polynomials

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Introduction

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Algebraic Circuit Complexity

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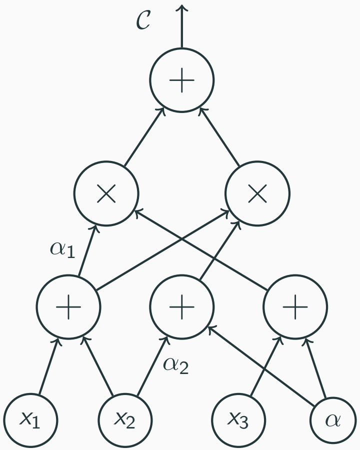
Given an algebraic circuit as input, how efficiently can one check if it computes the identically zero polynommial?

Part 1: Lower Bounds in Algebraic Circuit Complexity

Algebraic Models of Computation



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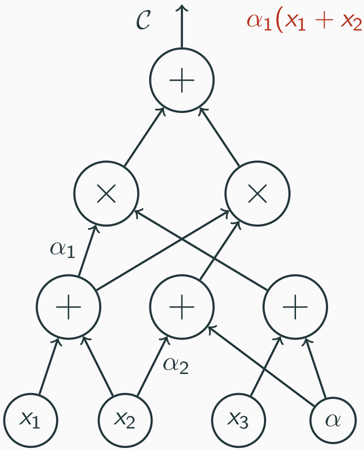

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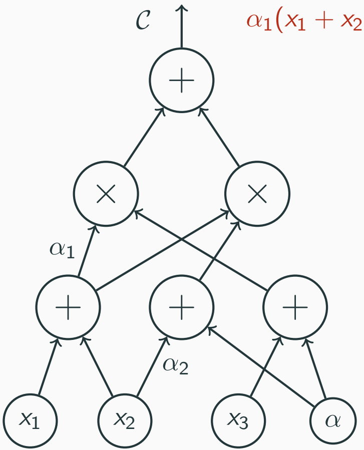

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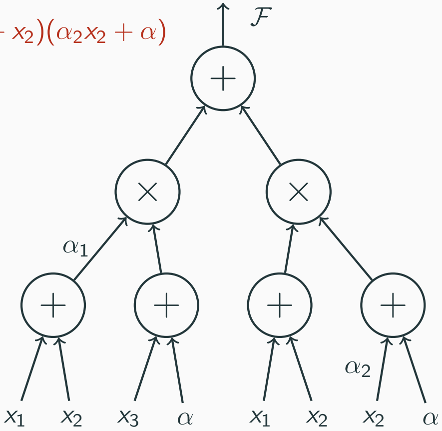
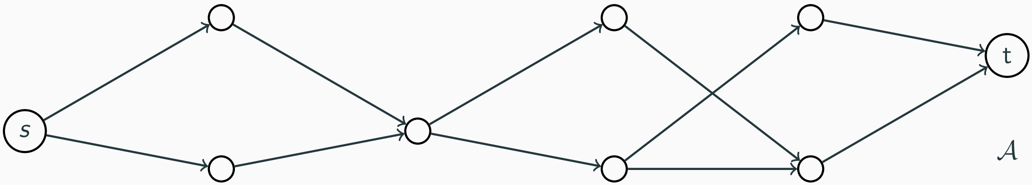


Algebraic Branching Programs



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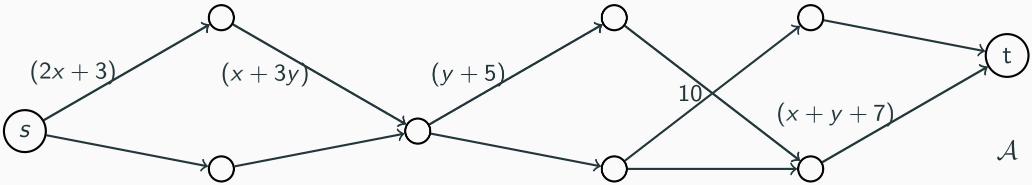

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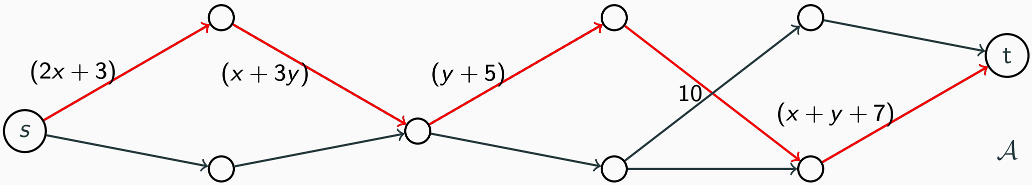

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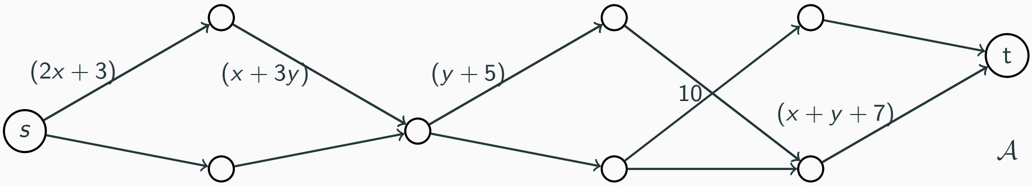

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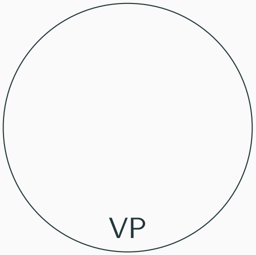

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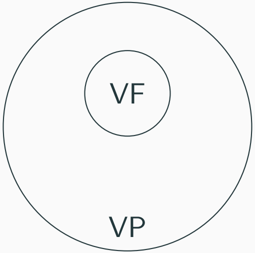

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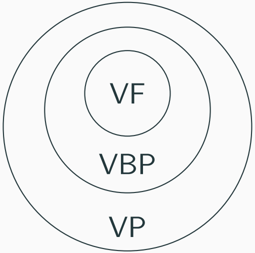

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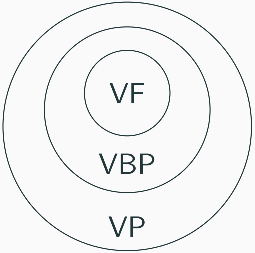

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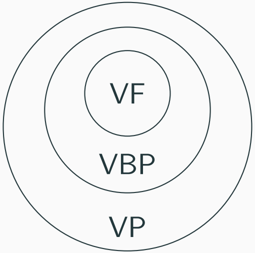

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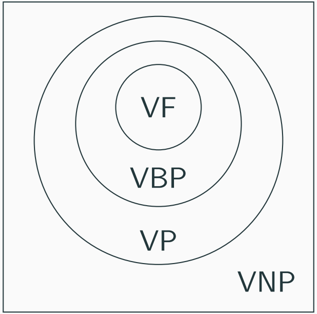
VNP: Explicit Polynomials



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3. The lower bound is also true for a multilinear polynomial

$$\mathrm{ESYM}_{n,0.1n}(\mathbf{x}) = \sum_{i_1 < \cdots < i_{0.1n} \in [n]} \sum_{j=1}^{n} x_{i_j}.$$

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2. Kalorkoti's method can not give a better bound against multilinear polynomials [Jukna].
3. Our result also shows a super-linear separation between the computational powers of circuits and formulas when computing multilinear polynomials.

Proof Overview: The ABP Lower Bound

Step 0 ([Kumar]): Look at the homogeneous case

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Questions?

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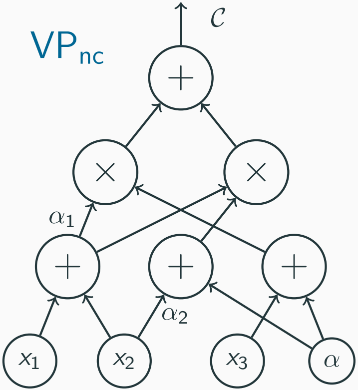

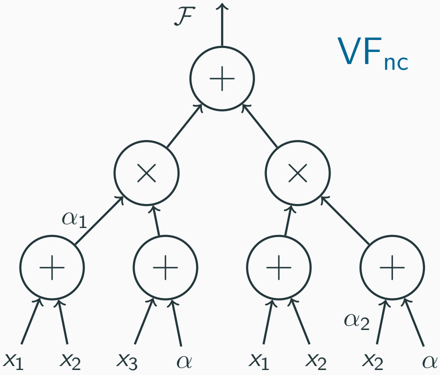

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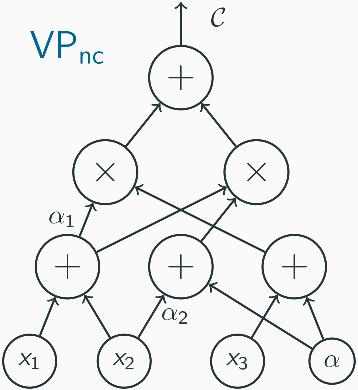

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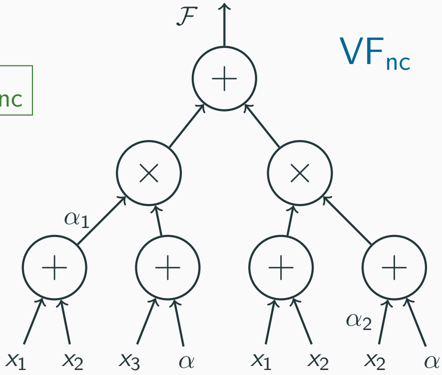

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So Nisan actually showed that $\mathsf{VBP}_{\mathsf{nc}} \neq \mathsf{VP}_{\mathsf{nc}}$.

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Main Result:

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Questions?

Part 2: Identity Testing and Algebraic Independence

Algebraic Independence

In the vector space $\mathbb{R}^3$ over $\mathbb{R}$,

$$(1, 0, 1) \qquad (0, 1, 0) \qquad (1, 2, 1)$$

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Algebraic Rank

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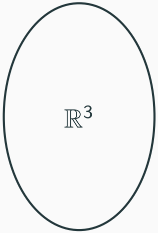
Algebraic Rank



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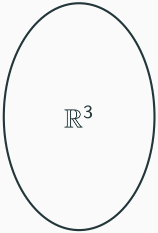

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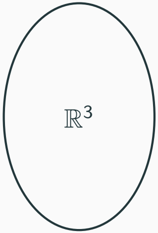

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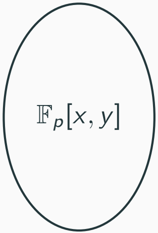

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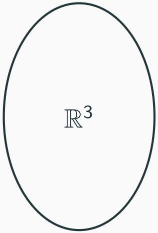

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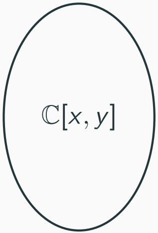

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Bonus: Helps in polynomial identity testing.

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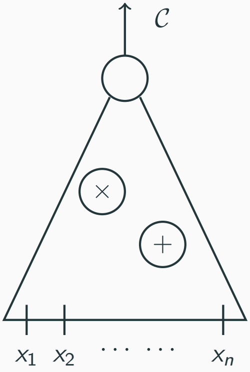
Given: Circuit $\mathcal{C}$ that computes an n-variate, degree d polynomial

Goal: Check whether $\mathcal{C} \cong$ Zero Polynomial.



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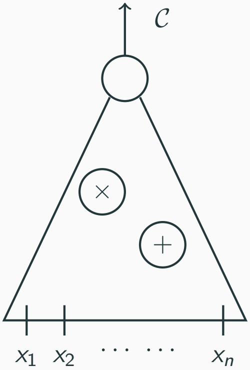

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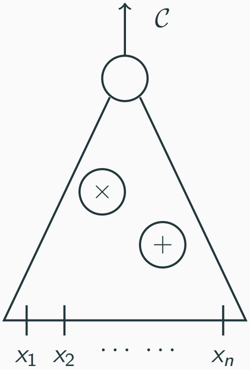

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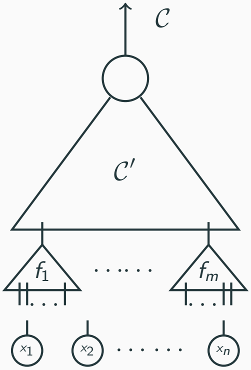

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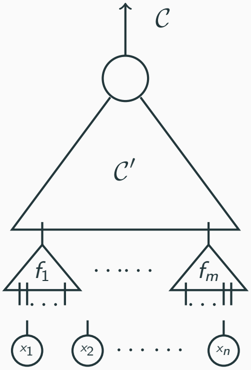

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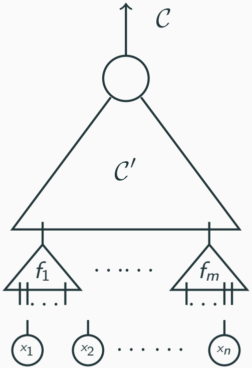
Faithful Maps & Polynomial Identity Testing



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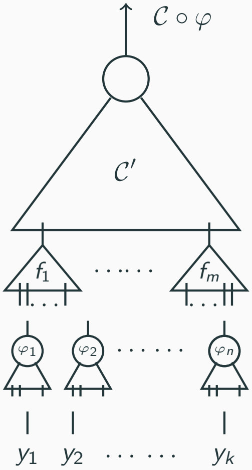

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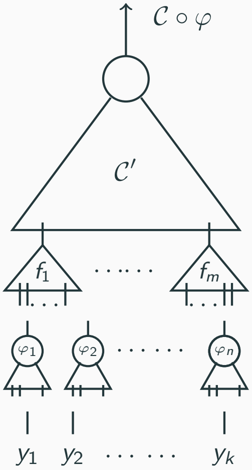

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The Jacobian Criterion [Jacobi]

If $\mathbb{F}$ has characteristic zero, the algebraic rank of $\{f_1, f_2, \ldots, f_m\}$ is equal to the linear rank of its Jacobian matrix.

Characteristic Zero Fields [B-M-S, A-S-S-S]

Step 1: Capture algebraic rank via linear rank of the Jacobian

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Alternate Criterion for the General Case [Pandey-Saxena-Sinhababu]

$f_1, f_2, \ldots, f_k \in \mathbb{F}[\mathbf{x}]$ are algebraically independent if and only if for every $(v_1, v_2, \ldots, v_k)$ with v_is in $\mathcal{I}_t$,

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where t is the inseparable degree of $\{f_1, f_2, \ldots, f_k\}$ and

$$\mathcal{I}_t = \langle \mathcal{H}_t(f_1), \mathcal{H}_t(f_2), \ldots, \mathcal{H}_t(f_k) \rangle_{\mathbb{F}(\mathbf{z})}^{\geq 2} \bmod \langle \mathbf{x} \rangle^{t+1} \subseteq \mathbb{F}(\mathbf{z})[\mathbf{x}].$$

Our Result

Suppose
- $\circ$ $f_1, \ldots, f_m \in \mathbb{F}[x_1, \ldots, x_n]$
- $\circ$ algebraic rank of $\{f_1, \ldots, f_m\} = k$
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whenever

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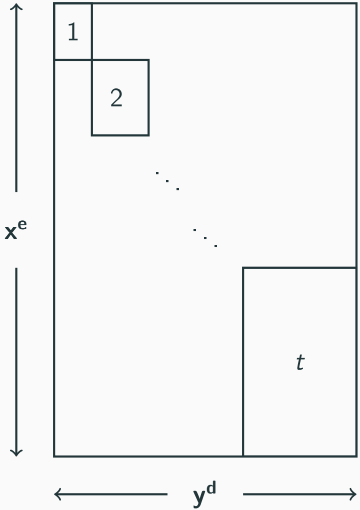
The Faithful Map



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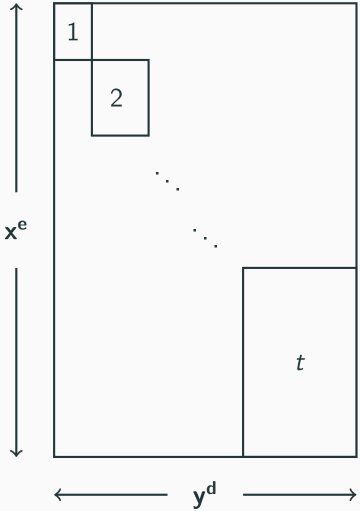

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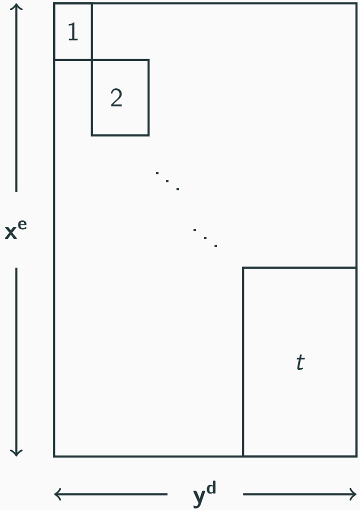

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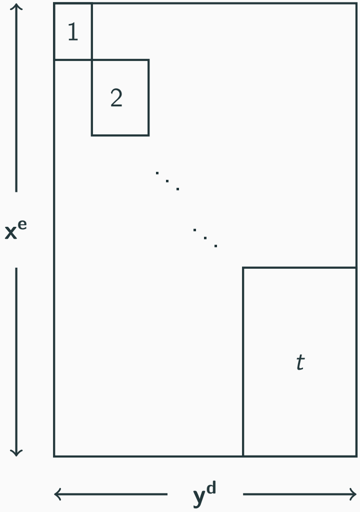

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$$\Phi(x_i) = a_i \cdot y_0 + \sum_{j \in [k]} s^{\mathrm{wt}(i)j} \cdot y_j$$

Questions?

Thankyou!