Constructing Faithful Maps over Arbitrary Fields

Prerona Chatterjee

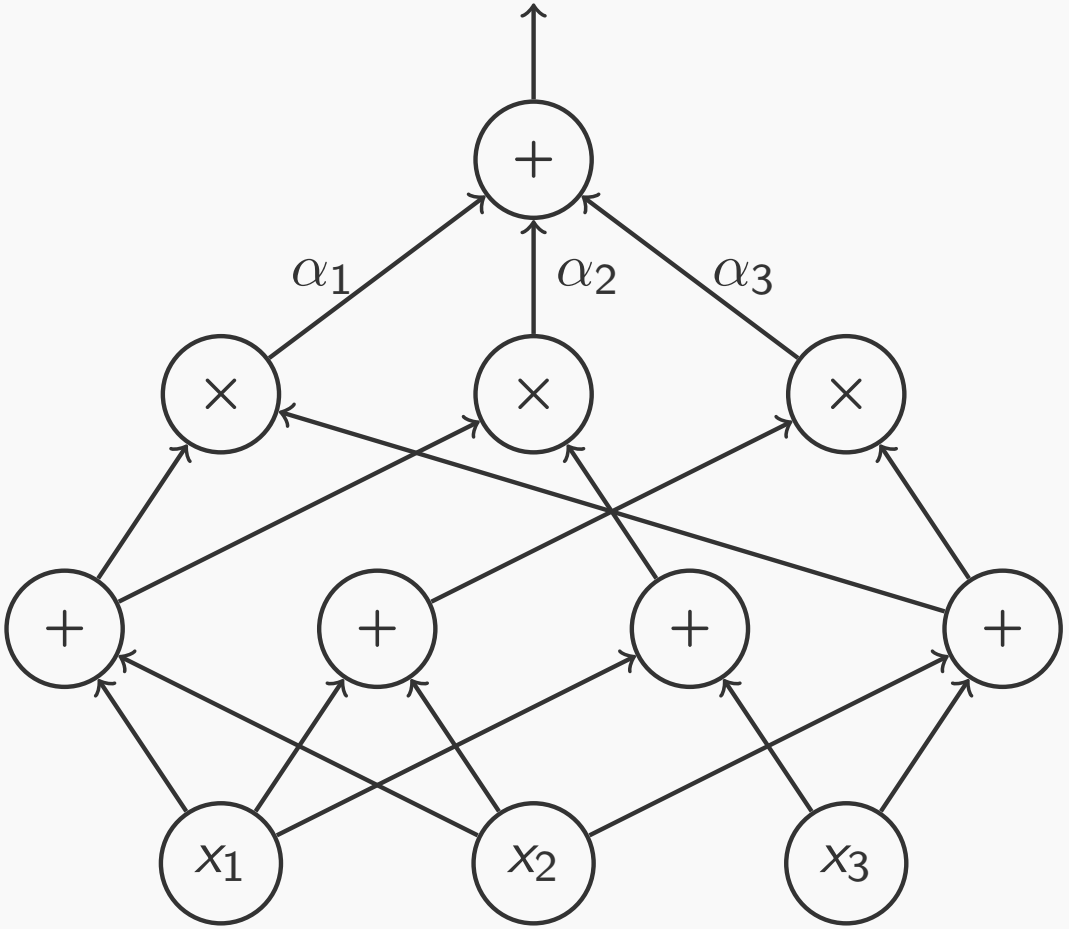
joint work with

Ramprasad Saptharishi

TIFR, Mumbai

February 16, 2018

Algebraic Circuits and Polynomial Identity Testing



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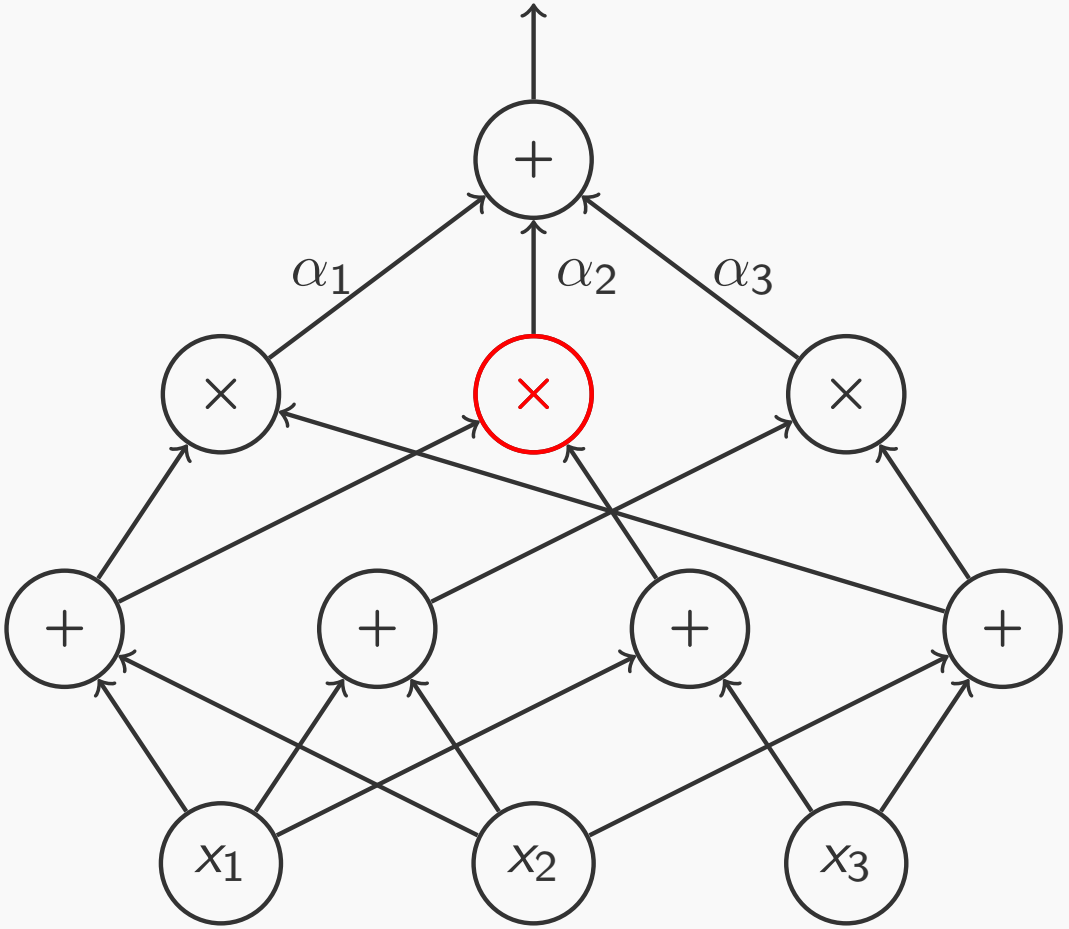

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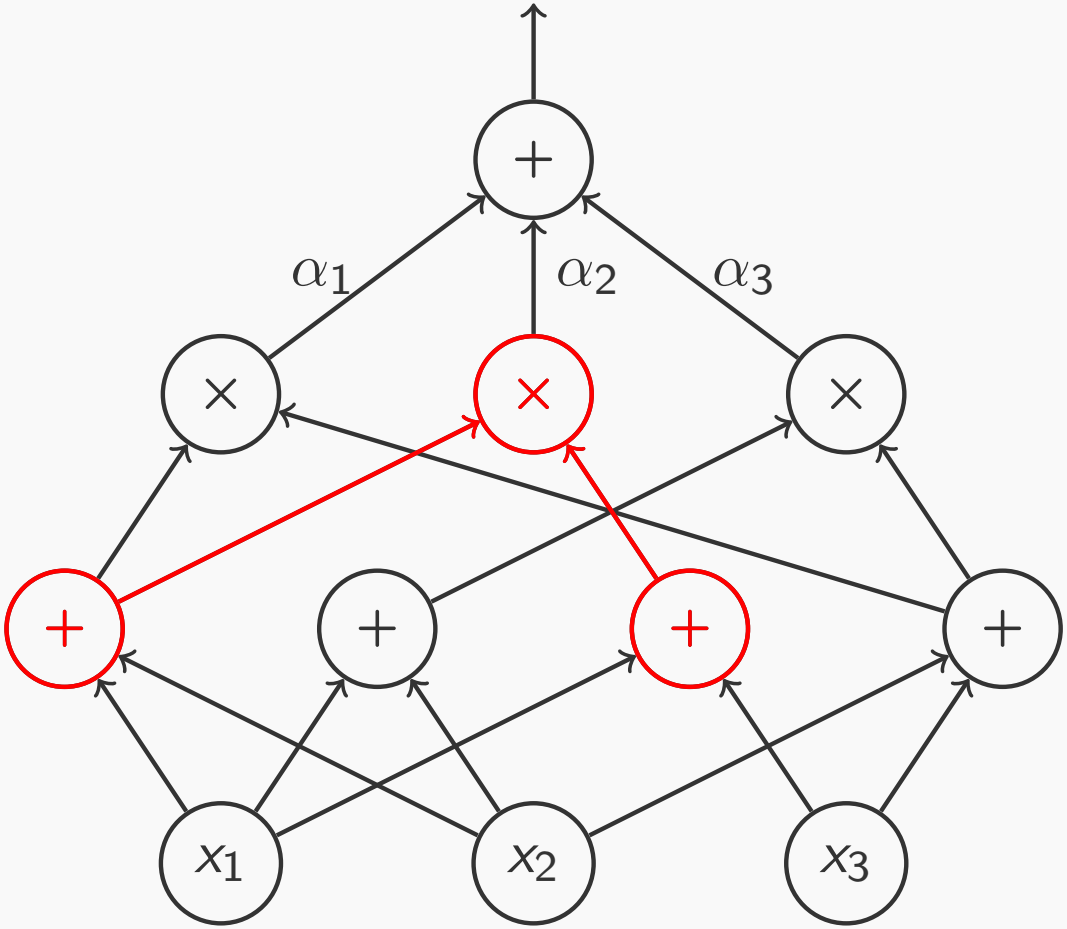
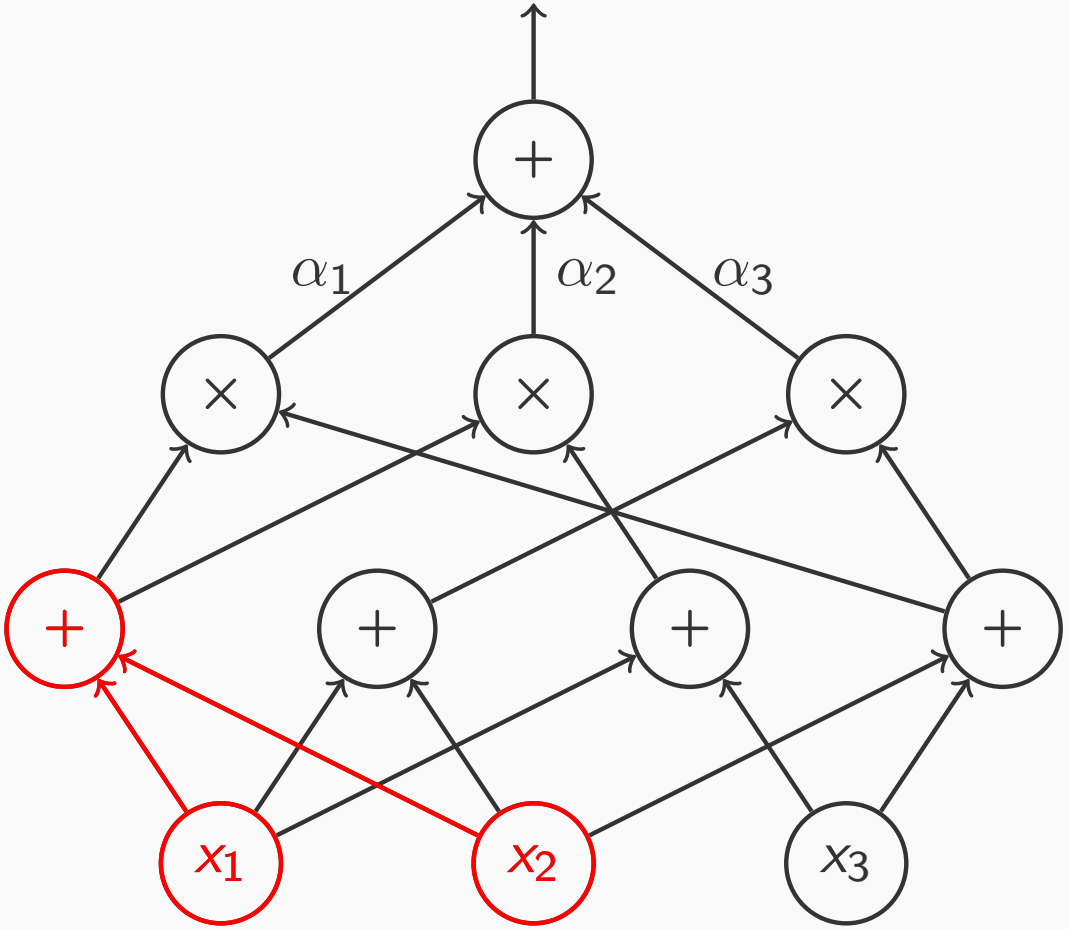


$$()\cdot()$$

Algebraic Circuits and Polynomial Identity Testing



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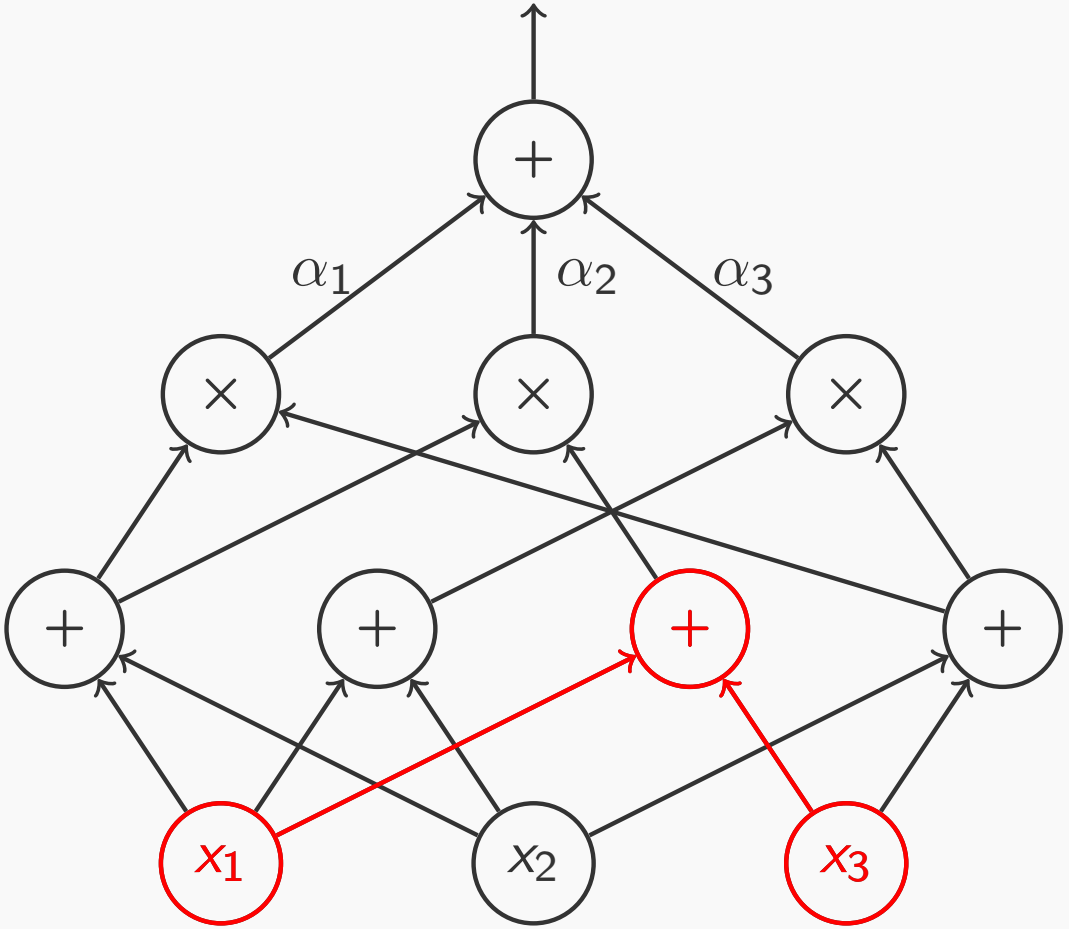

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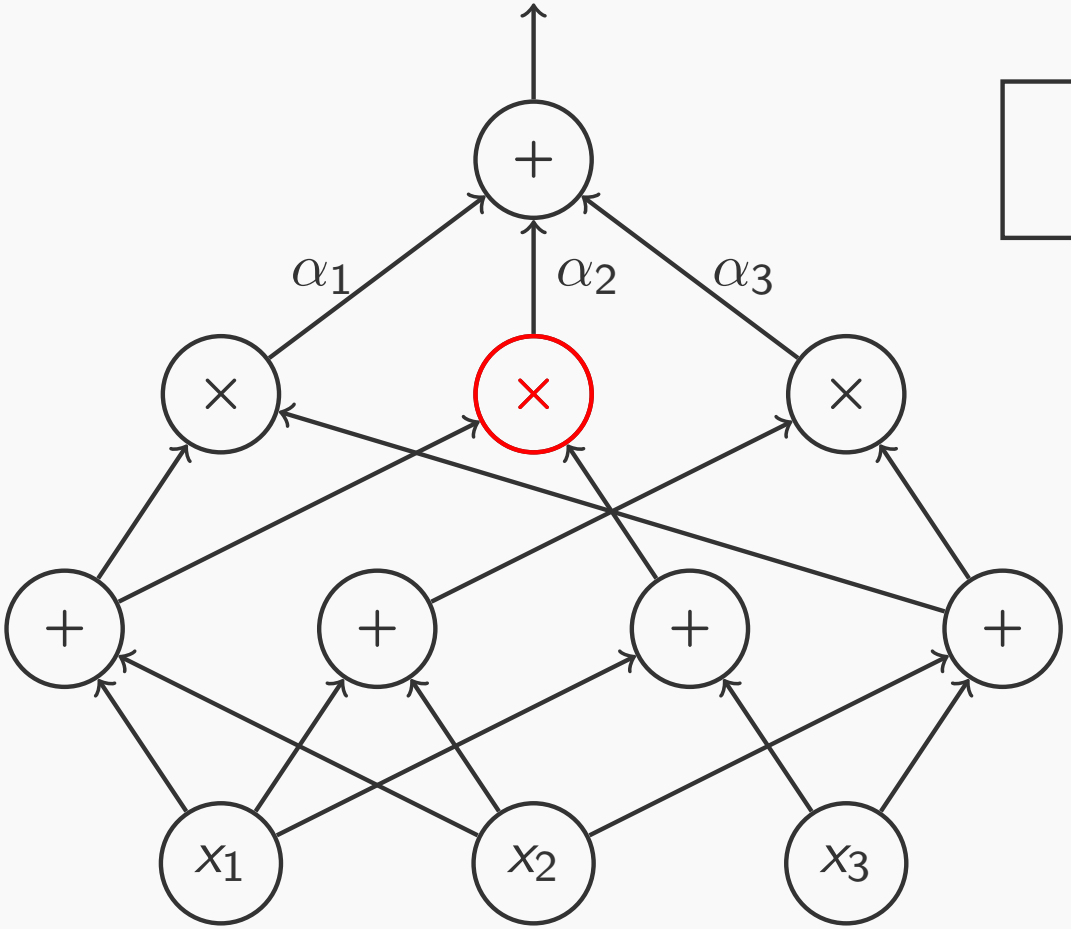

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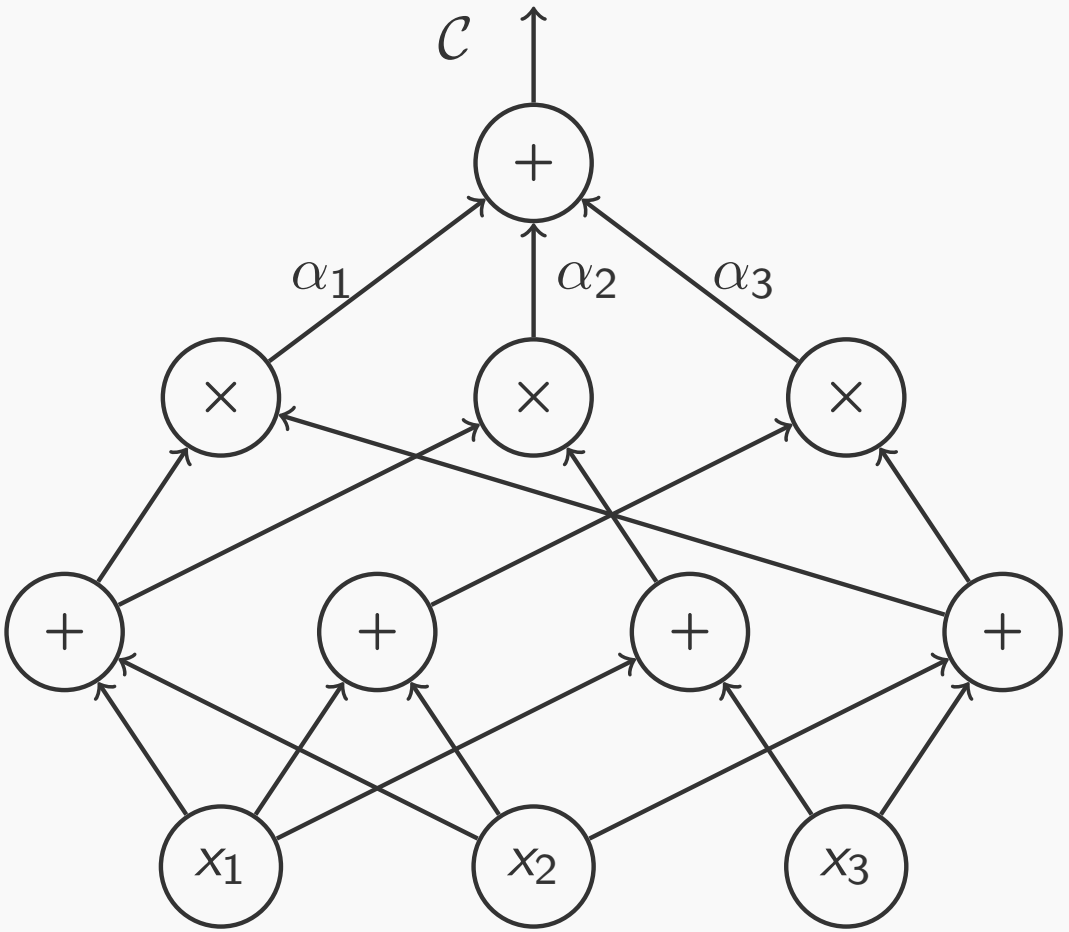
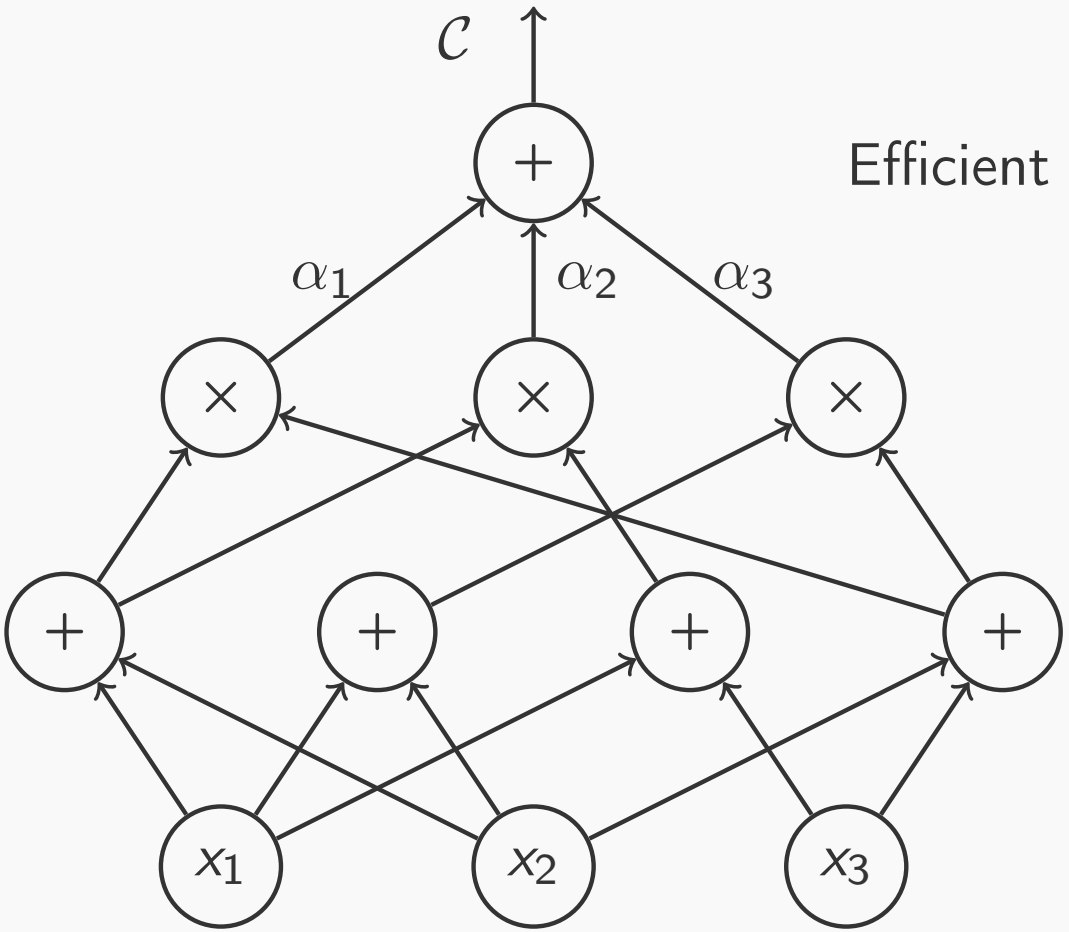


$$\mathcal{C} \stackrel{?}{=} 0$$

Algebraic Circuits and Polynomial Identity Testing



Efficient deterministic PIT not known for general circuits

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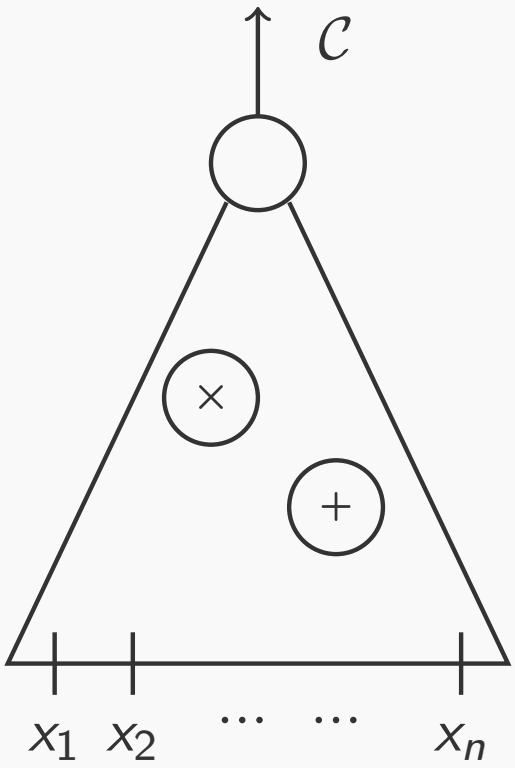
Polynomial Identity Testing and Faithful Maps



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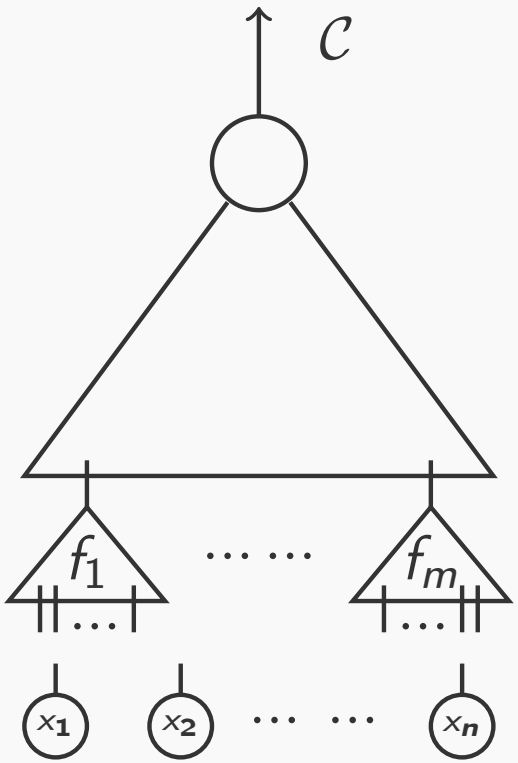
Polynomial Identity Testing and Faithful Maps



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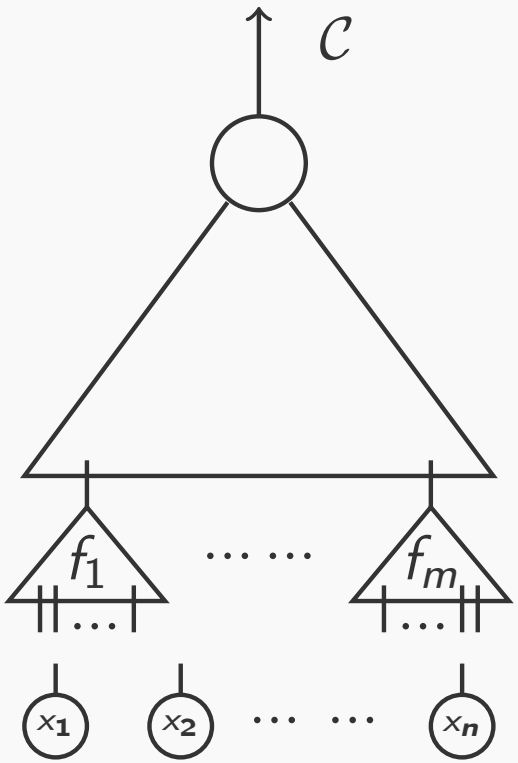

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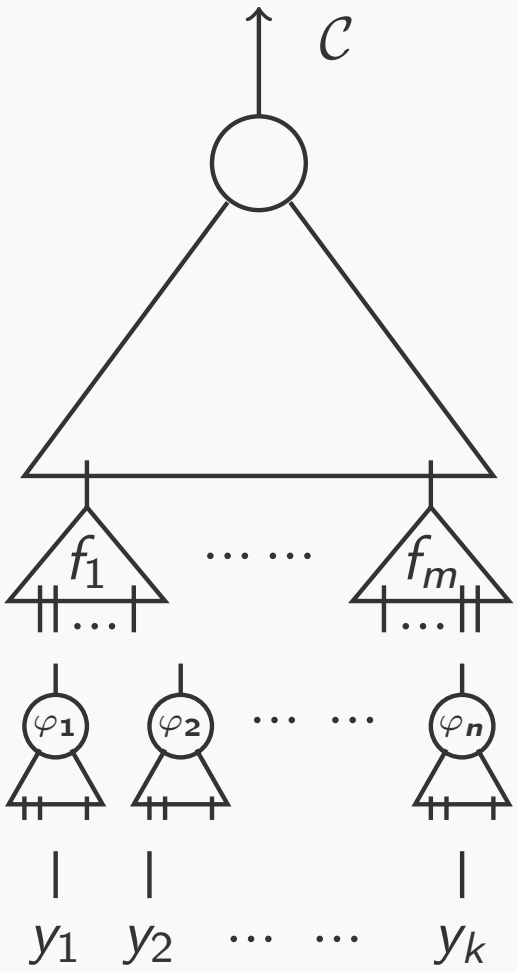

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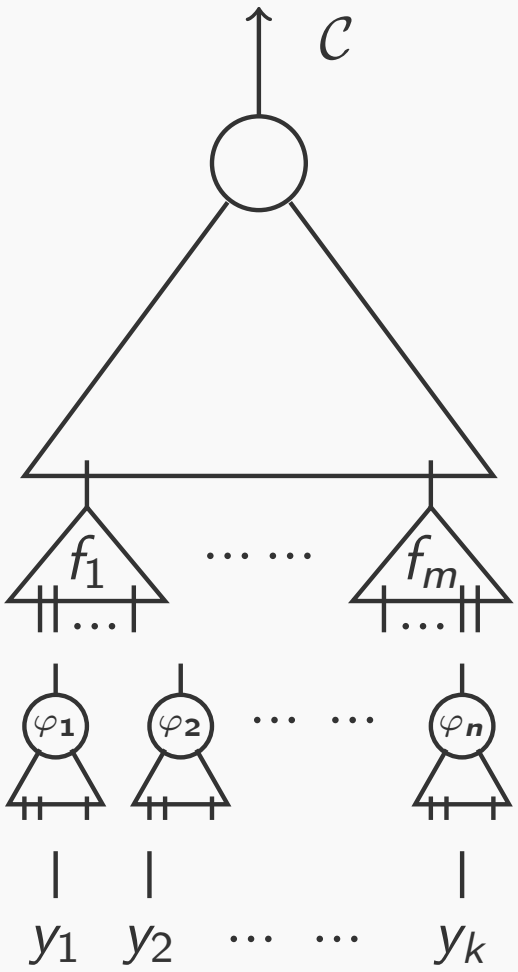

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$\mathcal{C}(f_1, f_2, \ldots, f_m) \neq 0$ if and only if

$(\mathcal{C}(f_1(\varphi), f_2(\varphi), \ldots f_m(\varphi))) \neq 0.$

How many are "Relevant"?

Definition: Algebraic Independence

A given set of polynomials $\{f_1, f_2, \ldots, f_m\} \subseteq \mathbb{F}[x_1, x_2, \ldots, x_n]$ is said to be algebraically dependent if there is a non-zero polynomial combination of these that is zero.

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For a set of polynomials $\{f_1, f_2, \ldots, f_m\}$, the family of all algebraically independent subsets form a matroid.

Thus, $\text{algrank}(f_1, f_2, \ldots, f_m)$ is well defined.

Checking for Relevance over Characteristic Zero fields

For $f_1, f_2, \ldots, f_m \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ and $\mathbf{f} = (f_1, f_2, \ldots, f_m)$,

$$\mathbf{J_x}(\mathbf{f}) = \begin{bmatrix} \partial_{x_1}(f_1) & \partial_{x_2}(f_1) & \cdots & \partial_{x_n}(f_1) \\ \partial_{x_1}(f_2) & \partial_{x_2}(f_2) & \cdots & \partial_{x_n}(f_2) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1}(f_m) & \partial_{x_2}(f_m) & \cdots & \partial_{x_n}(f_m) \end{bmatrix}$$

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The Jacobian Criterion

If $\mathbb{F}$ has characteristic zero, $\{f_1, f_2, \ldots, f_m\}$ is algebraically independent if and only if its Jacobian matrix is full rank.

"Good" Maps

Definition: Faithful Maps

Given a set of polynomials $\{f_1, f_2, \ldots, f_m\}$ with algebraic rank k, a map

$$\varphi : \{x_1, x_2, \ldots, x_n\} \to \mathbb{F}(y_1, y_2, \ldots, y_k)$$

is said to be a faithful map if the algebraic rank of $\{f_1(\varphi), f_2(\varphi), \ldots, f_m(\varphi)\}$ is also k.

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The Connection [BMS11, ASSS12]: Given a set of polynomials $\{f_1, f_2, \ldots, f_m\}$ and a faithful map φ; for any circuit $\mathcal{C}(z_1, \ldots, z_m)$,

$$\mathcal{C}(f_1, f_2, \ldots, f_m) \neq 0 \Leftrightarrow (\mathcal{C}(f_1(\varphi), f_2(\varphi), \ldots f_m(\varphi))) \neq 0.$$

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What we need: φ such that

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For every $m \times n$ matrix A, $\text{rank}(A) = \text{rank}(AM_\varphi)$.

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Family of matrices or one matrix parameterised by s: $\left\{M_{\varphi(s)}\right\}_{s \in \mathcal{F}}$

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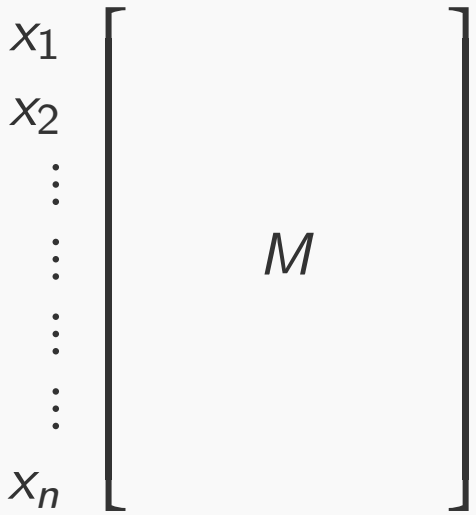

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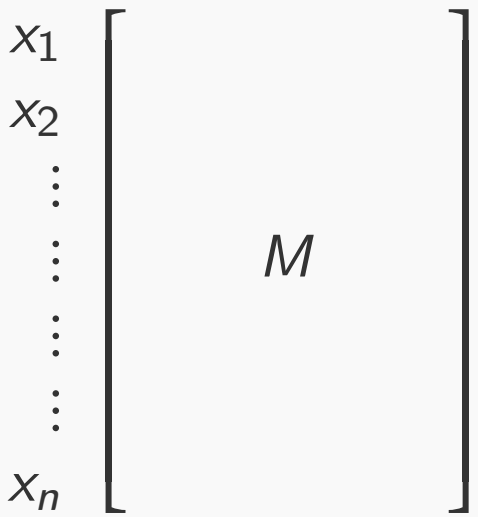

Binet-Cauchy:

$$\det(AM) = \sum_{B \subseteq \{x_i\},\ |B|=k} \det(A_B)\det(M_B).$$

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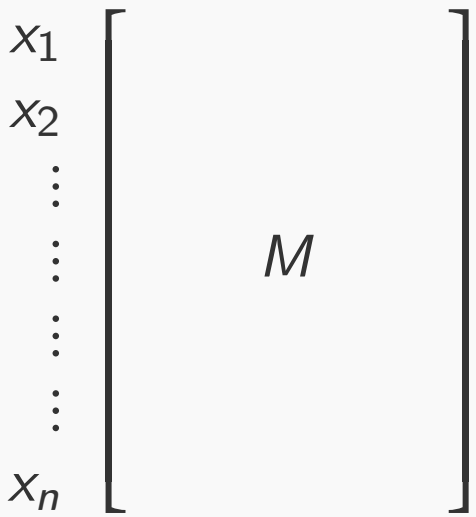

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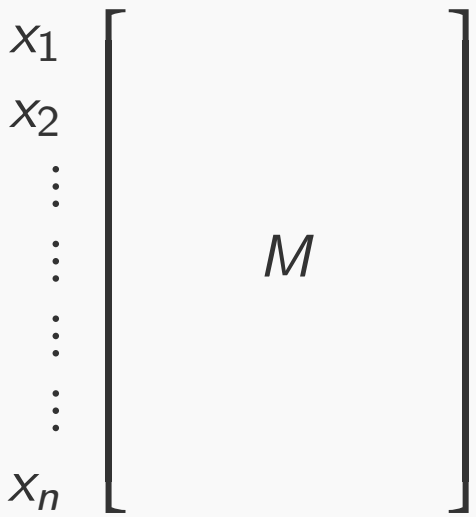

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$$\begin{matrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{matrix} \begin{bmatrix} s^{\mathrm{wt}(1)} \\ s^{\mathrm{wt}(2)} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ s^{\mathrm{wt}(n)} \end{bmatrix}$$

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$$A_x(\alpha, \beta, \gamma) = A'_x(\alpha^{p^{k_1}}, \beta, \gamma), A_y(\alpha, \beta, \gamma) = A'_y(\alpha^{p^{k_2}}, \beta, \gamma)$$

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$$\mathbf{J}_{x,y} = \begin{bmatrix} y^{p-1} & (p-1)xy^{p-2} \\ (p-1)x^{p-2}y & x^{p-1} \end{bmatrix}$$

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Reason: $(x, f_1, f_2) : A_x(\alpha, \beta, \gamma) \qquad (y, f_1, f_2) : A_y(\alpha, \beta, \gamma)$

$$\partial_\alpha(A_x) = 0 = \partial_\alpha(A_y)$$

$$A_x(\alpha, \beta, \gamma) = A'_x(\alpha^{p^{k_1}}, \beta, \gamma), A_y(\alpha, \beta, \gamma) = A'_y(\alpha^{p^{k_2}}, \beta, \gamma)$$

For $k = \max\{k_1, k_2\}$, p^k : Inseparable degree of $\{f_1, f_2\}$.

Hasse derivatives

For any $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ and $\mathbf{z} \in \mathbb{F}^n$,

$$f(\mathbf{x}+\mathbf{z}) - f(\mathbf{z}) = \underbrace{x_1 \cdot \partial_{x_1} f + \cdots + x_n \cdot \partial_{x_n} f}_{\text{Jacobian}} + \text{higher order terms}$$

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In general, the Hasse derivative of f with respect to $\mathbf{x}^\mathbf{e}$ is the coefficient of $\mathbf{x}^\mathbf{e}$ in $f(\mathbf{x}+\mathbf{z}) - f(\mathbf{z})$.

The Criterion over Arbitrary fields

Definition: A new Operator

For any $f \in \mathbb{F}[x_1, x_2, \ldots, x_n]$,

$$\mathcal{H}_t(f) = \deg^{\leq t}\left(f(\mathbf{x}+\mathbf{z}) - f(\mathbf{z})\right)$$

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$$\hat{\mathcal{H}}(\mathbf{f}) = \begin{bmatrix} \ldots & \mathcal{H}_t(f_1) & \ldots \\ \ldots & \mathcal{H}_t(f_2) & \ldots \\ & \vdots & \\ \ldots & \mathcal{H}_t(f_k) & \ldots \end{bmatrix}.$$

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The [PSS16] Criterion

A given set of polynomials $\{f_1, f_2, \ldots, f_k\} \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ is algebraically independent if and only if for a random $\mathbf{z} \in \mathbb{F}^n$, $\{\mathcal{H}_t(f_1), \mathcal{H}_t(f_2), \ldots, \mathcal{H}_t(f_k)\}$ are linearly independent in

$$\frac{\mathbb{F}(\mathbf{z})[x_1, x_2, \ldots, x_n]}{\mathcal{I}_t}$$

where t is the inseparable degree of $\{f_1, f_2, \ldots, f_k\}$ and $\mathcal{I}_t$ is some fixed ideal of $\mathbb{F}(\mathbf{z})[x_1, x_2, \ldots, x_n]$.

Alternate Statement for the [PSS16] criterion

$\{f_1, f_2, \ldots, f_k\}$ is algebraically independent if and only if for every $(v_1, v_2, \ldots, v_k)$ with v_is in $\mathcal{I}_t$,

$$\mathcal{H}(\mathbf{f}, \mathbf{v}) = \begin{bmatrix} \ldots & \mathcal{H}_t(f_1) + v_1 & \ldots \\ \ldots & \mathcal{H}_t(f_2) + v_2 & \ldots \\ & \vdots & \\ \ldots & \mathcal{H}_t(f_k) + v_k & \ldots \end{bmatrix} \text{ has full rank over } \mathbb{F}(\mathbf{z}).$$

What we want to show

$$\mathcal{H}(\mathbf{f}(\varphi), \mathbf{u}) = \begin{bmatrix} \dots & \mathcal{H}_t(f_1(\varphi)) + u_1 & \dots \\ \dots & \mathcal{H}_t(f_2(\varphi)) + u_2 & \dots \\ & \vdots & \\ \dots & \mathcal{H}_t(f_k(\varphi)) + u_k & \dots \end{bmatrix}$$

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$$\varphi : x_i \to \sum_{j=1}^{k} s_{ij} y_j + a_i y_0 \text{ and } z_i \to \sum_{j=1}^{k} s_{ij} w_j + a_i w_0$$

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M_φ preserves rank

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where

$$M_\varphi(\mathbf{x}^{\mathbf{e}}, \mathbf{y}^{\mathbf{d}}) = \begin{cases} \mathrm{coeff}_{\mathbf{y}^{\mathbf{d}}}(\varphi(\mathbf{x}^{\mathbf{e}})) & \text{if } \sum e_i = \sum d_i \\ 0 & \text{otherwise} \end{cases}$$

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Goal: Make each block Vandermonde type.

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Extend wt to sub-matrices cleverly.

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By Binet-Cauchy formula,

$$\det(\varphi(\mathcal{H}(\mathbf{f}, \mathbf{v})) \cdot M'_\varphi) \neq 0.$$

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Size bounds: $p = O(n^{3t})$, $s = O(p)$.

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Choice of a: Depends on the model under consideration.

Applications to PIT

Concept of faithful maps used to solve PIT in following settings:

1. $\mathcal{C}(f_1, f_2, \ldots, f_m)$, where f_i are sparse polynomials.
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More to come....

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Thank you!!

References I

Manindra Agrawal, Chandan Saha, Ramprasad Saptharishi, and Nitin Saxena.
Jacobian hits circuits: hitting-sets, lower bounds for depth-d occur-k formulas & depth-3 transcendence degree-k circuits.
In *Proceedings of the 44th Symposium on Theory of Computing Conference, STOC 2012, New York, NY, USA, May 19 - 22, 2012*, pages 599–614, 2012.

Malte Beecken, Johannes Mittmann, and Nitin Saxena.
Algebraic independence and blackbox identity testing.
CoRR, abs/1102.2789, 2011.

Ariel Gabizon and Ran Raz.
Deterministic extractors for affine sources over large fields.
In *46th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2005), 23-25 October 2005, Pittsburgh, PA, USA, Proceedings*, pages 407–418, 2005.

References II

Anurag Pandey, Nitin Saxena, and Amit Sinhababu.
Algebraic independence over positive characteristic: New criterion and applications to locally low algebraic rank circuits.
In *41st International Symposium on Mathematical Foundations of Computer Science, MFCS 2016, August 22-26, 2016 - Kraków, Poland*, pages 74:1–74:15, 2016.