A Quadratic Lowerbound against Algebraic Branching Programs

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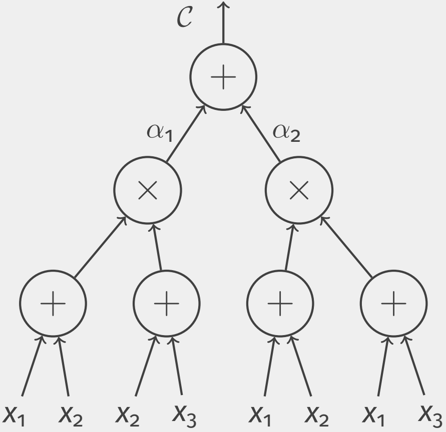
Algebraic Formulas and Algebraic Circuits



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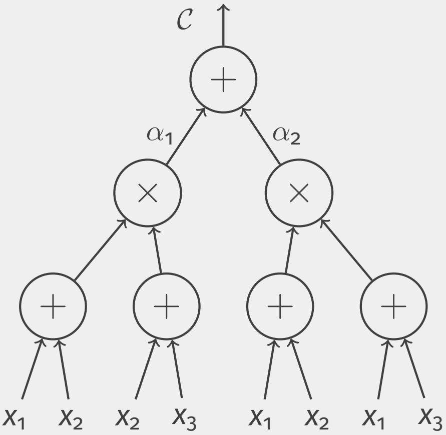

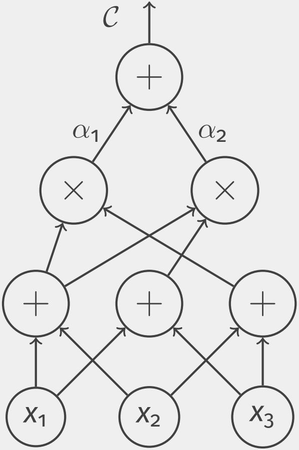

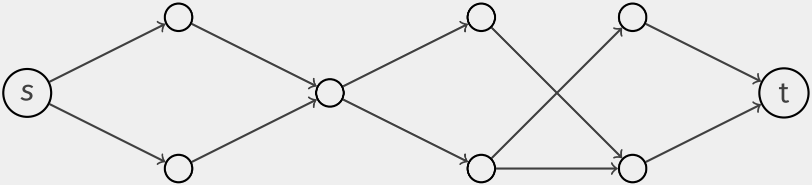
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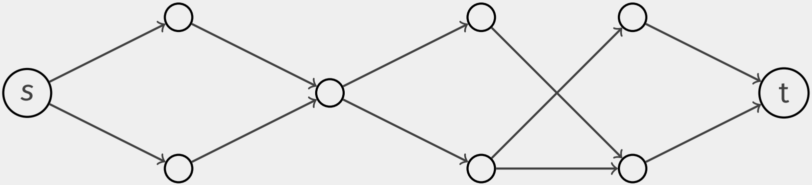
Algebraic Branching Programs



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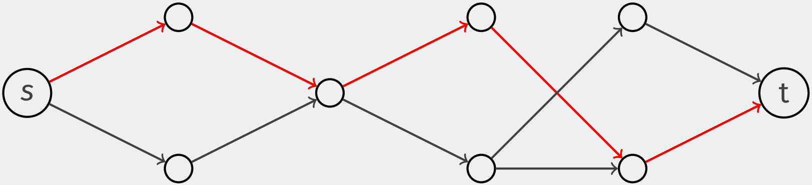

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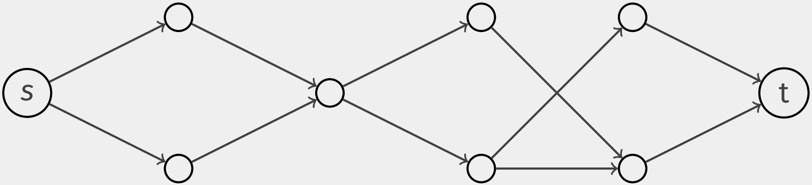

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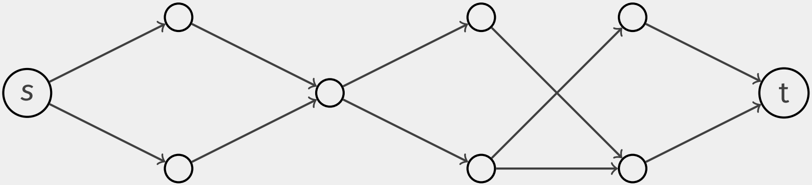

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VF $\subseteq$ VBP $\subseteq$ VP

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Any fan-in 2 circuit computing $\sum_{i=1}^{n} x_i^d$ requires $\Omega(n \log d)$ multiplication gates.

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Note: For an n-variate individual degree d polynomial, a lowerbound of $\Omega(nd)$ is trivial for formulas.

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Our Main Result

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- Partition the vertices according to the layer it is in.
- Say $V_k = \{v_1, \ldots, v_{t_k}\}$ are the vertices in the k-th layer.
- Show that $t_k \geq n/2$ for every $k \in [d-1]$.

Base Case: Proof Overview

$P_{n,d} = \sum_{i=1}^{n} x_i^d$

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Note: $\deg(g'_i), \deg(h'_i) \leq [d-1]$ for every $i \in [d-1]$.

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For $R = \sum_{i=1}^{t_k} (\alpha_i \cdot h_i' + \beta_i \cdot g_i' + \alpha_i \cdot \beta_i)$,

$$P_{n,d} = \left(\sum_{i=1}^{t_k} g_i' \cdot h_i' \right) + R$$

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$$P_{n,d} = \left(\sum_{i=1}^{t_k} g_i' \cdot h_i' \right) + R \implies P_{n,d} - R = \sum_{i=1}^{t_k} g_i' \cdot h_i'$$

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where $\deg(R) \leq d - 1$.

Base Case: Proof Overview (contd.)

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Base Case: Proof Overview (contd.)

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$$\dim(\mathcal{V}) \geq n - 2t_k \implies n - 2t_k \leq 0 \implies t_k \geq n/2$$

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$$f = \sum_{i=1}^{n} x_i^d + \sum_{i=1}^{r} A_i(\mathbf{x}) \cdot B_i(\mathbf{x}) + \delta \quad \text{where} \quad A_i(0) = 0 = B_i(0)$$

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- **Updated Base Case:** $V_k = \{v_1, \ldots, v_{t_k}\}$ — k-th layer vertices

$$\text{For every } k \in [d-1], \quad t_k \geq n/2 - r$$

The Main Claim

ℓ-th step

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no. of error terms = $r_{\ell+1} \leq r_\ell + \frac{s_\ell}{d_\ell/3}$

Completing the Proof

No. of steps needed = L

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$$s_0 \leq c \cdot nd \qquad d_{\ell+1} \leq \frac{2}{3} d_\ell \implies \frac{1}{d_\ell} \leq \frac{2}{3} \cdot \frac{1}{d_{\ell+1}}$$

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$$s_0 \leq c \cdot nd \qquad d_{\ell+1} \leq \frac{2}{3} d_\ell \implies \frac{1}{d_\ell} \leq \frac{2}{3} \cdot \frac{1}{d_{\ell+1}}$$

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Completing the Proof

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$$s_0 \le c \cdot nd \qquad d_{\ell+1} \le \frac{2}{3} d_\ell \implies \frac{1}{d_\ell} \le \frac{2}{3} \cdot \frac{1}{d_{\ell+1}}$$

$$\begin{aligned} r_L &= 3 \cdot s_0 \left(\frac{1}{d_0} + \frac{1}{d_1} \cdots + \frac{1}{d_{L-1}} \right) \\ &\le \frac{3 \cdot c \cdot nd}{d_{L-1}} \left(\left(\frac{2}{3}\right)^{L-1} + \left(\frac{2}{3}\right)^{L-2} + \cdots + 1 \right) < 9 \cdot c \cdot n \end{aligned}$$

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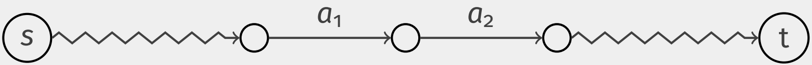
The Main Claim (once again...)

ℓ-th step

Given: ABP $\mathcal{A}_\ell$ of size = s_ℓ
no. of layers = d_ℓ
no. of error terms = r_ℓ

Want to construct: ABP $\mathcal{A}_{\ell+1}$ of size = $s_{\ell+1} \leq s_\ell$
no. of layers = $d_{\ell+1} \leq \frac{2}{3} d_\ell$
no. of error terms = $r_{\ell+1} \leq r_\ell + \frac{s_\ell}{d_\ell/3}$

Proving the Claim



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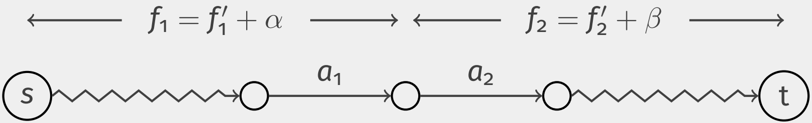

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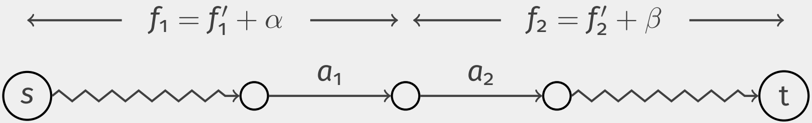
$\mathcal{A} = f_1 \cdot f_2$



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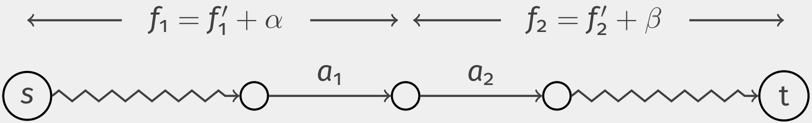

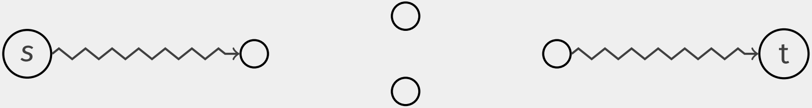

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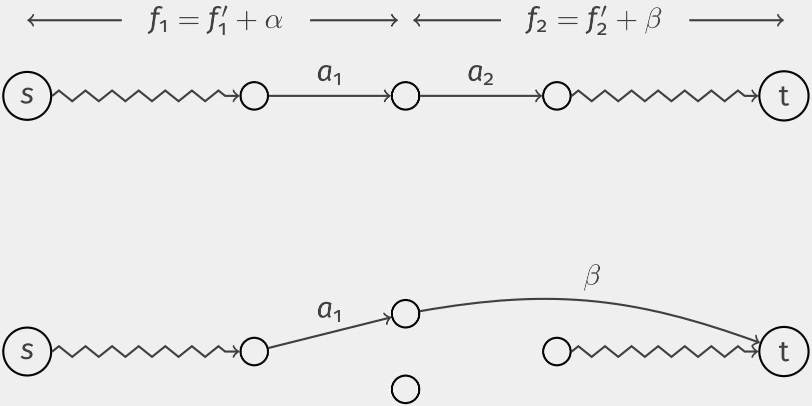

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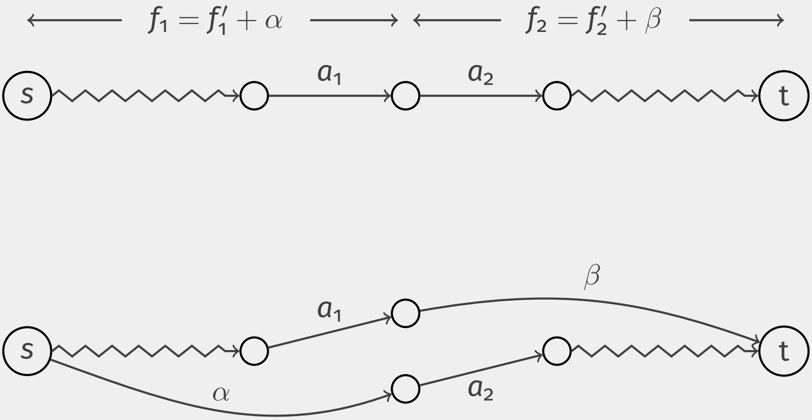

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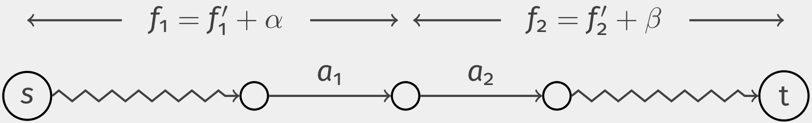
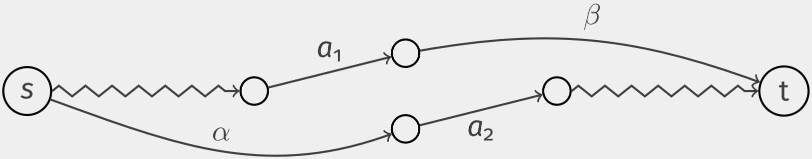


$\mathcal{A}' = \beta \cdot f_1 + \alpha \cdot f_2$



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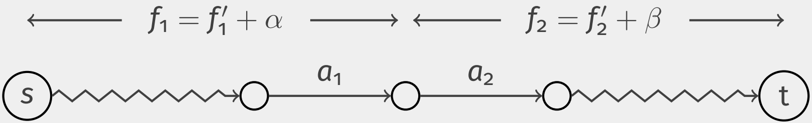

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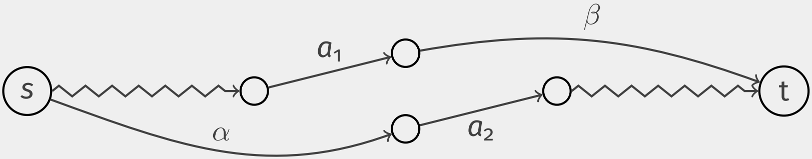

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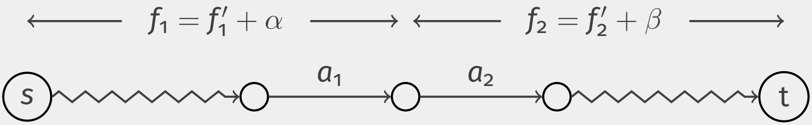

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Thank you!