Separating ABPs and Structured Formulas in the Non-Commutative Setting

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March 19, 2021

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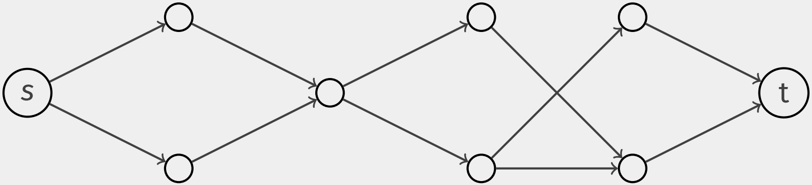
- Hardness Amplification is known (**CILM ’18**).

Algebraic Branching Programs



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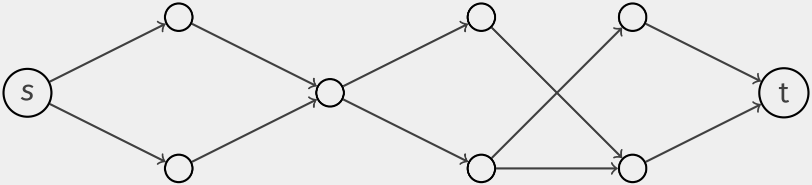

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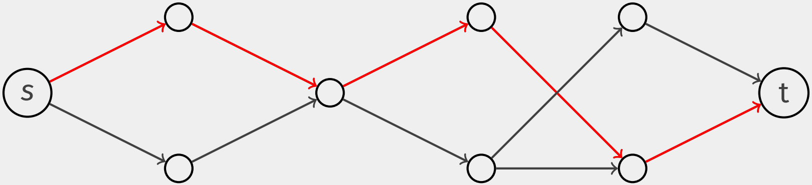

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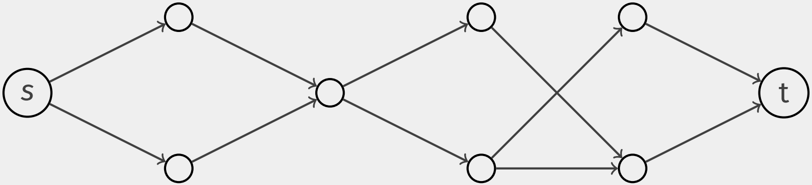

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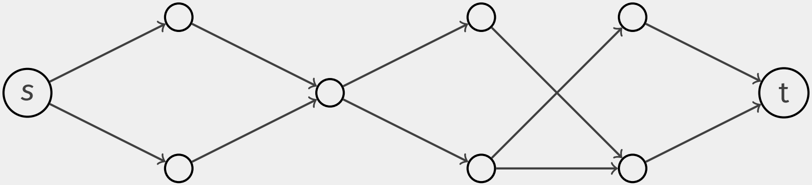

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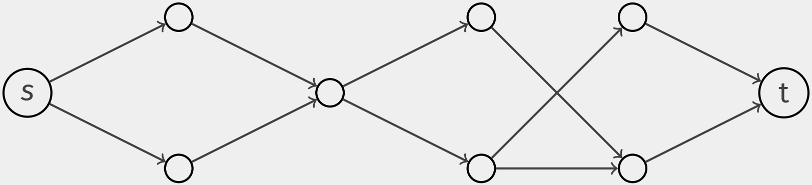

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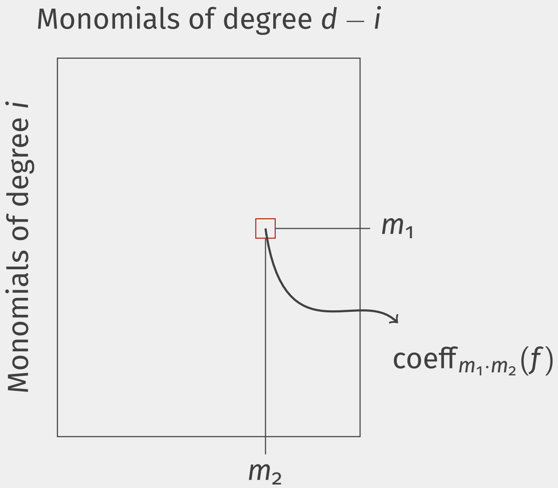
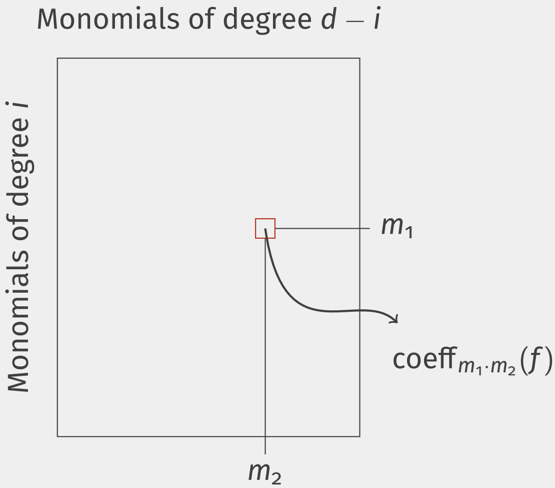
For a general polynomial f of degree d, $f = \text{Hom}_0(f) + \text{Hom}_1(f) + \cdots + \text{Hom}_d(f)$.



f is a polynomial of degree d.

For every $1 \leq i \leq d$, consider the matrix $M_f(i)$ described alongside.



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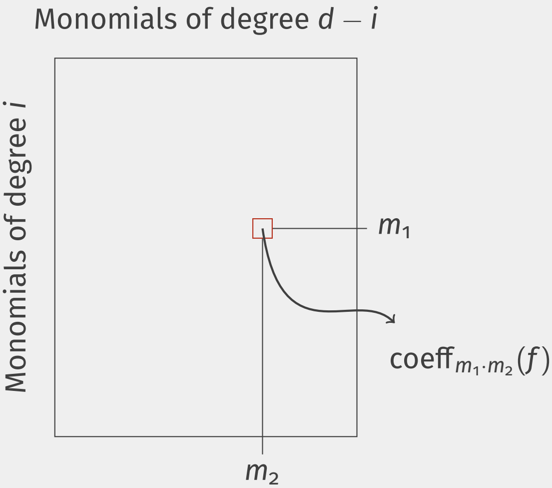
Nisan's Characterisation



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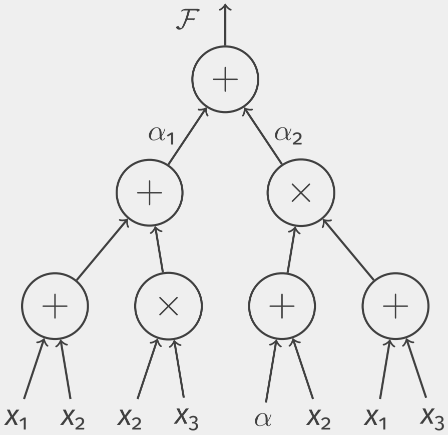
If $\mathcal{A}$ is the smallest ABP computing f,

$$\text{size}(\mathcal{A}) = \sum_{i=1}^{d} \text{rank}(M_f(i)).$$

Algebraic Formulas



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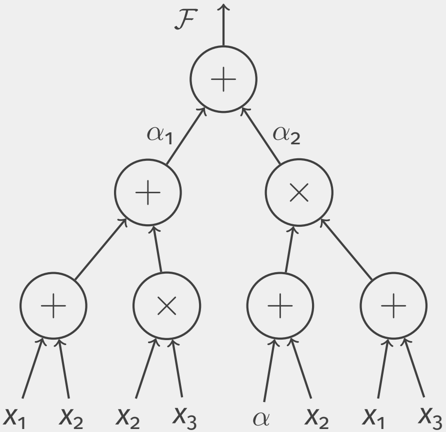

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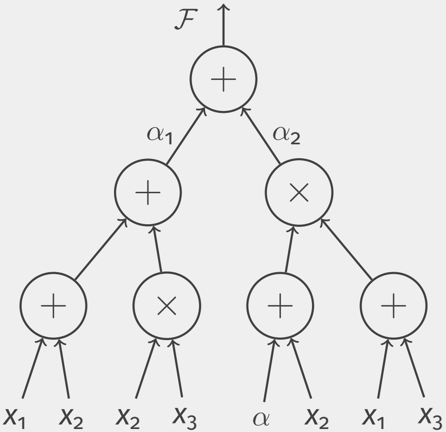
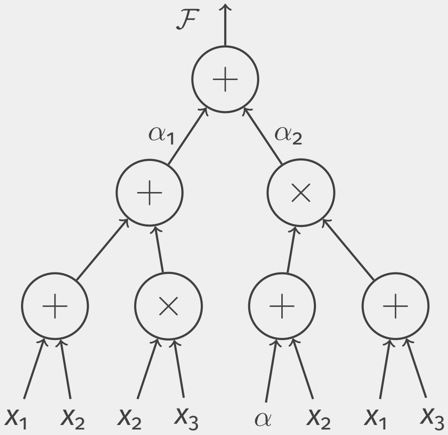


$$\mathsf{VF}_{\mathsf{nc}} \subseteq \mathsf{VBP}_{\mathsf{nc}}$$

Super-polynomial ABP lower bounds

$\Downarrow$

super-polynomial formula lower bounds.



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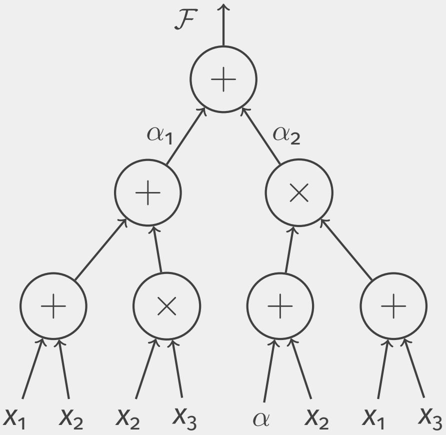
Any ABP (or formula) computing $\mathrm{Det}_n(\mathbf{x})$ has size at least $2^{\Omega(n)}$.



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Question: Is $\mathsf{VF}_{\mathsf{nc}} = \mathsf{VBP}_{\mathsf{nc}}$?

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$$f(\mathbf{x}) \xrightarrow[\text{using some fixed } \sigma \in S_n]{\text{Order the monomials}} f^{(nc)}(\mathbf{x})$$

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Note:

$$\mathrm{ESYM}^{(\mathrm{ord})}_{n,d} = \sum_{1\le i_1<\ldots<i_d\le n} x^{(1)}_{i_1} \cdots x^{(d)}_{i_d}$$

is abecedarian w.r.t. both $\left\{X_k = \left\{x_i^{(k)}\right\}_{i\in[n]}\right\}_{k\in[d]}$ as well as $\left\{X_i = \left\{x_i^{(k)}\right\}_{k\in[d]}\right\}_{i\in[n]}$.

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f is the polynomial computed between (u,a) and $(v,b) \implies f = f[a,b+1)$.

What was known and what we show

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Possible New Approaches to Solving the General Question

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A positive answer to either of these questions would imply that $\mathsf{VBP}_{nc} \neq \mathsf{VF}_{nc}$.

The Explicit Statement

$$\text{linked\_}\mathrm{CHSYM}_{n,d}(\mathbf{x}) = \sum_{i_0=1}^{n} \left(\sum_{i_0 \leq i_1 \leq \ldots \leq i_d \leq n} x_{i_0,i_1} \cdot x_{i_1,i_2} \cdots x_{i_{d-1},i_d} \right)$$

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- Abecedarian with respect to $\{X_i \,:\, 1 \le i \le n\}$ where $X_i = \left\{x_{ij} \,:\, 1 \le j \le n\right\}$.
- $\text{linked\_}\mathrm{CHSYM}_{n,d}(\mathbf{x})$ can be computed by an abecedarian ABP of size $O(nd)$.
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- A super-polynomial lower bound against abecedarian formulas for linked_$\mathrm{CHSYM}_{\log n,n}(\mathbf{x})$ would imply that $\mathrm{VF}_{\mathrm{nc}} \neq \mathrm{VBP}_{\mathrm{nc}}$.

The Structured ABP Upper Bound

$$h_{n,d}(\mathbf{x}) = \text{linked\_}\mathrm{CHSYM}_{n,d}(\mathbf{x}) = \sum_{i_0=1}^{n} \left(\sum_{i_0 \le i_1 \le \ldots \le i_d \le n} x_{i_0,i_1} \cdot x_{i_1,i_2} \cdots x_{i_{d-1},i_d} \right)$$

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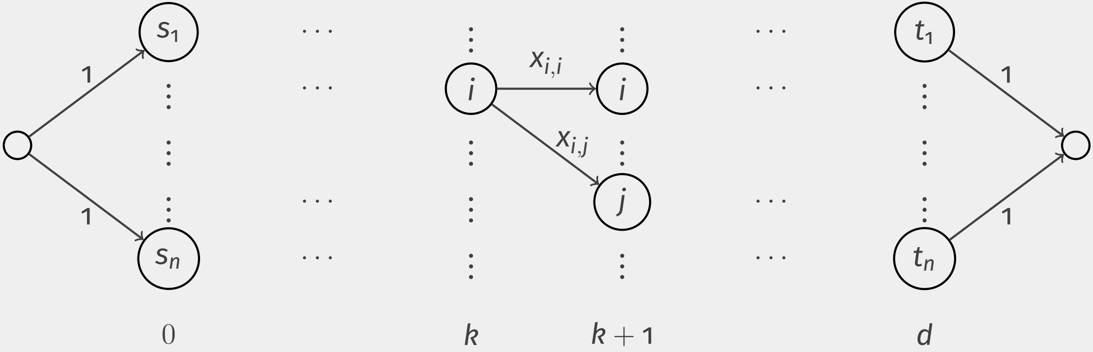

Proof Idea of the Structured Formula Lower Bound

Let $\mathcal{F}'$ be an abecedarian formula of size $s = O(n^{\varepsilon \log \log n})$ computing $h_{n/2, \log n}(\mathbf{x})$.

1. Homogenise to get $\mathcal{F}'_1$ of size $\mathrm{poly}(s)$.

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Proof Idea of the Structured Formula Lower Bound

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6. Using $\mathcal{F}_1$ construct a homogeneous multilinear formula of size $O(n^{c\cdot\varepsilon \log n})$ computing $\mathrm{ESYM}_{n,n/2}(\mathbf{x})$.
7. Choose ε such that Step 6 contradicts the known lower bound against this model for $\mathrm{ESYM}_{n,n/2}(\mathbf{x})$.

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A little about Step 1

- Brent's Depth Reduction proof works in the non-commutative setting as well.
- This allows Raz's Homogenisation proof to go through in this setting as well.
- It also answers a question by Nisan.

  $D(f)$: Depth Complexity; $F(f)$: Formula Complexity

  $$\text{Is } D(f) \leq O(\log F(f))?$$

We answer the question in the positive.

Proof Idea for Converting Formulas into Abecedarian Ones

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  the polynomial at $(v,(a,b))$ is $f_v[a,b]$.

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- When $m = O(\log s)$, the size of $\mathcal{F}'$ remains $\text{poly}(s)$.

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$\Longrightarrow \quad \mathsf{VF}_{\mathsf{nc}} \neq \mathsf{VBP}_{\mathsf{nc}}$

Thank you!