Compactness Theorem for Propositional Logic:

A Topological Proof

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Introduction

How far is it from computer science to topology? Surprisingly, it is not far at all.

This presentation brings forward the closeness of topology to computer science by proving the The Compactness Theorem of Propositional Logic using Topological Methods, which following Tarski gives the theorem its name.

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Importance

Compactness is one of the central notions of logic - as Jerome Keisler put it, The most useful theorem in model theory is probably the compactness theorem (1965, p. 113) - yet it has received relatively little philosophical attention.

The Compactness Theorem for Propositional Logic

Let $\mathcal{P}$ be a set of propositional constants, and $\mathcal{W}$ be an infinite set of formulas over $\mathcal{P}$.
If every finite subset of $\mathcal{W}$ is satisfiable, so is $\mathcal{W}$.

Proposition

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Truth Value of a Proposition

If a proposition is true, then we say its truth value is true, and if a proposition is false, we say its truth value is false.

Logical Connectives

- $\neg$ denotes 'not' or negation: If P is a proposition constant, then $\neg$P is true if P is false, and is false if P is true.
- $\vee$ denotes 'or' or disjunction: If P, Q are propositional constants, then P $\vee$ Q is false if both P and Q are false, and true otherwise.
- $\wedge$ denotes 'and' or conjunction: If P, Q are propositional constants, then P $\wedge$ Q is true if both P and Q are true, and false otherwise.
- $\rightarrow$ denotes 'conditional' or implication: If P and Q are propositional constants, the conditional statement P $\rightarrow$ Q is false when P is true and Q is false, and is true otherwise.
- $\leftrightarrow$ denotes 'biconditional': If P and Q are propositional constants, the biconditional statement P $\leftrightarrow$ Q, is true if P and Q have the same truth values, and false otherwise.

Well-Formed Formula over a set of Propositional Constants

Let $\mathcal{P}$ be a set of propositional constants. We define Well-Formed formulas (or simply Formula) over $\mathcal{P}$ as follows:

- P is a formula $\forall P \in \mathcal{P}$.
- If ϕ is a formula, then $\neg\phi$ is also a formula.
- If ϕ_1 and ϕ_2 are formulas, then $(\phi_1 \vee \phi_2)$, $(\phi_1 \wedge \phi_2)$, $(\phi_1 \rightarrow \phi_2)$ and $(\phi_1 \leftrightarrow \phi_2)$ are also formulas.
- Nothing else is a formula.

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- Nothing else is a formula.

Note

A formula can only contain a finite sequence of propositional constants from $\mathcal{P}$ even though $\mathcal{P}$ can be arbitrary.

Truth Assignments

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A truth assignment is a function $f : \mathcal{P} \to \{True, False\}$.

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Models

Let $\mathcal{P}$ be a set of propositional constants.
A model for $\mathcal{P}$ is a function that assigns to each P $\in \mathcal{P}$ a proposition (i.e. meaning).
We can also view a model as a truth assignment because ultimately we work only with the truth values of the proposition corresponding to the propositional constants and not the proposition itself.

Satisfiablity

Let $\mathcal{P}$ be a set of propositional constants and let $\mathcal{W}$ be a set of formulas over $\mathcal{P}$.

- $\phi \in \mathcal{W}$ is satisfiable if it is possible to find a truth assignment that makes the formula true.
- $\mathcal{W}$ is satisfiable if it is possible to find a truth assignment that makes ϕ true $\forall \phi \in \mathcal{W}$.
- $\mathcal{W}$ is finitely satisfiable if given any finite $\mathcal{W}' \subseteq \mathcal{W}$, $\mathcal{W}'$ is satisfiable.

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Validity

Let $\mathcal{P}$ be a set of propositional constants, and let ϕ be a formula over $\mathcal{P}$.

- ϕ is a tautology if all interpretations make ϕ true.
- ϕ is a contradiction if no interpretation makes ϕ true.

The Compactness Theorem for Propositional Logic

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What we need and what we have...

What we need: A truth assignment that makes ϕ true $\forall \phi \in \mathcal{W}$.

What we have: A truth assignment that makes ϕ true $\forall \phi \in \mathcal{W}'$ given any finite $\mathcal{W}' \subseteq \mathcal{W}$

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If we try and recall the definition of Compactness in Topology, we realise that we have something very similar:
Given a topological space $(X, \mathcal{T})$, X is said to be compact iff each of its open covers has a finite sub-cover or alternately:

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Given a topological space $(X, \mathcal{T})$, X is said to be compact iff for every collection of closed subsets $\mathcal{C}$ of X, any finite $\mathcal{C}' \subseteq \mathcal{C}$ has non empty intersection $\Rightarrow \mathcal{C}$ has non empty intersection.

Searching for the correct Topological Space

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- As we need a truth assignment that makes ϕ true $\forall \phi \in \mathcal{W}$, $\mathcal{C}$ must be a collection of the sets $\{X_\phi\}_{\phi \in \mathcal{W}}$ where $X_\phi =$ The set of all truth assignments that make ϕ true

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- As X_ϕ must be members of the topology we will be considering, each X_ϕ must be a subset of the set we start with. Hence, the set we should consider is the set of all truth assignments on $\mathcal{P}$

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- As X_ϕ must be members of the topology we will be considering, each X_ϕ must be a subset of the set we start with. Hence, the set we should consider is the set of all truth assignments on $\mathcal{P}$
- The topological space constructed must be compact
- The topological space must be such that X_ϕ is closed $\forall \phi \in \mathcal{W}$

Defn: Topology on a Set

Let X be a set. A topology on X is a collection $\mathcal{T}$ of subsets of X satisfying:

- $\mathcal{T}$ contains $\emptyset$ and X
- $\mathcal{T}$ is closed under arbitrary unions
  (i.e. if $U_i \in \mathcal{T}$ $\forall i \in I$ for an arbitrary indexing set I, then $\bigcup_{i \in I} U_i \in \mathcal{T}$)
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Remark

One may view the first condition as a special case of the other two since $\emptyset$ is a union of the empty collection and X is the intersection of the empty (hence finite) collection.

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Defn: Open Sets, Closed Sets and Neighbouhoods

Now, let $(X,\mathcal{T})$ be a Topological Space.

- The elements of $\mathcal{T}$ are called open subsets of X.
- A subset of X is said to be closed if its complement in X is open.
- A neighbourhood of p is a set $U \in \mathcal{T}$ with $p \in U$.

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Defn: Basis for a Topological Space

Let (X,$\mathcal{T}$) be a Topological Space.
$\mathcal{B}$ is called a basis for $\mathcal{T}$ if $\forall$G $\in \mathcal{T}$, G can be written as an arbitrary union of elements of $\mathcal{B}$.

Defn: Basis for some Topology on a Set

If only a set X is given, a basis for some topology on X is a collection $\mathcal{B}$ of subsets of X (called basis elements) satisfying the following properties:

- For each $x \in X$, $\exists\ B \in \mathcal{B}$ such that $x \in B$.
- Let $B_1, B_2 \in \mathcal{B}$. If $x \in B_1 \cap B_2$, then $\exists\ B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq B_1 \cap B_2$.

Note: The collection of all arbitrary unions of members of $\mathcal{B}$ forms a topology on X and is denoted by $\mathcal{T}_{\mathcal{B}}$

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Defn: Sub-basis for some topology on a Set

If X is a set, any collection S of subsets of X form a sub-basis for some topology on X.
The topology induced by S is the arbitrary union of finite intersections of members of S.

Defining the Set

Let $\mathcal{P}$ be a non-empty set of propositional constants and let $\mathcal{W}$ be the set of all formulas over $\mathcal{P}$. Consider the space of truth assignments, $\mathcal{V} = \{v : \mathcal{P} \to \{True, False\}\}$.

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For each $v \in \mathcal{V}$, define a function $\overline{v} : \mathcal{W} \to \{True, False\}$ by

$$\overline{v}(\phi) = \begin{cases} True & v \text{ makes } \phi \text{ true} \\ False & v \text{ makes } \phi \text{ false} \end{cases}$$

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For each $\phi \in \mathcal{W}$, define $X_\phi = \{v \in \mathcal{V} : \overline{v}(\phi) = True\}$.

Basis for a topology on the Set

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Proof of the fact that $\mathcal{B}$ is a basis

- As $\mathcal{P} \neq \emptyset$, $\exists$ P $\in \mathcal{P}$. Define $\phi = P \vee \neg P$. Then $\phi \in \mathcal{W}$ is a tautology $\Rightarrow X_\phi = \mathcal{V} \Rightarrow \{X_\phi : \phi \in \mathcal{W}\}$ covers $\mathcal{V}$.

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  Then $\phi \in \mathcal{W}$ is a tautology
  $\Rightarrow X_\phi = \mathcal{V} \Rightarrow \{X_\phi : \phi \in \mathcal{W}\}$ covers $\mathcal{V}$.
- Let $\alpha, \beta \in \mathcal{W}$ and let $v \in X_\alpha \cap X_\beta$.
  Then, $\overline{v}(\alpha), \overline{v}(\beta) = \textit{True}$.
  Consider $\gamma = \alpha \wedge \beta$.
  Then $\overline{v}(\gamma) = \textit{True}$ and so, $v \in X_\gamma$.
  Moreover, $\forall\ v' \in X_\gamma$, $\overline{v'}(\gamma) = \textit{True}$
  $\Rightarrow \overline{v'}(\alpha), \overline{v'}(\beta) = \textit{True}$
  $\Rightarrow v' \in X_\alpha \cap X_\beta$
  and so, $X_\gamma \subseteq X_\alpha \cap X_\beta$.

Defn: Continuous map

Let (X, $\mathcal{T}$) and (Y, $\mathcal{T}'$) be two topological spaces, and f: X $\rightarrow$ Y be a map. f is said to be a continuous map if any one of the following hold:

- f^{-1}(G) $\in \mathcal{T}$ $\forall$G $\in$ (Y, $\mathcal{T}'$)
- f^{-1}(F) is closed in $\mathcal{T}$ $\forall$F closed in (Y, $\mathcal{T}'$)
- f^{-1}(B) $\in \mathcal{T}$ $\forall$ basic open sets B, of (Y, $\mathcal{T}'$)
- f^{-1}(S) $\in \mathcal{T}$ $\forall$ sub-basic open sets S of (Y, $\mathcal{T}'$)

Defn: Open and Closed Maps

Let (X, $\mathcal{T}$) and (Y, $\mathcal{T}'$) be two topological spaces, and $f\colon X \to Y$ be a map.

- f is said to be an open map if $f(B) \in \mathcal{T}'$ $\forall$ basic open set B of (X, $\mathcal{T}$).
- f is said to be a closed map if $f(F)$ is closed in $\mathcal{T}'$ $\forall$F closed in (X, $\mathcal{T}$).

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Defn: Homeomorphism

Let (X, $\mathcal{T}$) and (Y, $\mathcal{T}'$) be two topological spaces, and f: X $\rightarrow$ Y be a map. f is said to be a homeomorphism map if:

- f is bijective
- f is continuous
- f is an open map (i.e. f^{-1} is a continuous map).

In this case, (X, $\mathcal{T}$) and (Y, $\mathcal{T}'$) are said to be homoemorphic.

Defn: Arbitrary Product of Sets

Let Λ be an arbitrary non-empty indexing set, and $(X_\alpha, \mathcal{T}_\alpha)$ be topological spaces $\forall \alpha \in \Lambda$.
Define $X = \prod_{\alpha\in\Lambda} X_\alpha = \{x : \Lambda \to \bigcup_{\alpha\in\Lambda} X_\alpha$, such that $x(\alpha) \in X_\alpha \ \forall \alpha \in \Lambda\}$.

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Defn: Projection Maps on the Product Set

Define $\pi_\beta : X \to (X_\beta, \mathcal{T}_\beta)$ to be the β^{th} projection map defined by $\pi_\beta(x) = x(\beta)$.

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Define $\pi_\beta : X \to (X_\beta, \mathcal{T}_\beta)$ to be the β^{th} projection map defined by $\pi_\beta(x) = x(\beta)$.

Defn: The Product Topology

The product topology $\mathcal{T}_p$ on X is the smallest topology on X which makes π_β continuous $\forall \beta \in \Lambda$.
So, $\mathcal{T}_p$ is a topology on X having $\{\pi_\beta^{-1}(G_\beta)$ such that $G_\beta \in \mathcal{T}_\beta\}$ as a sub-basis.
Thus a basis for $\mathcal{T}_p$ is $\prod_{\alpha \in \Lambda} U_\alpha$ where U_α are members of $\mathcal{T}_\alpha$ $\forall \alpha \in \Lambda$ and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in \Lambda$.

Result: Projection Maps are Continuous

By the very definition of the topology on X, π_β is continuous $\forall \beta \in \Lambda$

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Defn: Continuity of Maps with Product Spaces as Co-Domain

Suppose:

- Λ be an arbitrary non-empty indexing set
- $(X_\alpha, \mathcal{T}_\alpha)$ are topological spaces $\forall \alpha \in \Lambda$ and $(X, \mathcal{T}_p)$ is the corresponding product topology
- $(Y, \mathcal{T})$ is any arbitrary topological space
- $f : (Y, \mathcal{T}) \to (X, \mathcal{T}_p)$ is a map
- $\pi_\alpha : X \to (X_\alpha, \mathcal{T}_\alpha)$ is the α^{th} projection map $\forall \alpha \in \Lambda$

Then, f is continuous iff $\pi_\alpha \circ f$ is continuous $\forall \alpha \in \Lambda$.

Defn: Literals, Simple Product and Minterms

Let $\mathcal{P}$ be a finite set of propositional constants.

- A propositional constant or its negation is called a literal.
- A simple product is a conjunction of literals.
- A minterm is a simple product in which for every statement variable, either the statement variable, or its negation appears exactly once.

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Defn: PDNF

Let $\mathcal{P}$ be a finite set of propositional constants.
If $\mathcal{W}$ is the set of all formulas over $\mathcal{P}$, a formula $\phi \in \mathcal{W}$ is said to be in PDNF if it is a finite disjunction of minterms.

Note:

Any formula $\phi \in \mathcal{W}$ other than a contradiction has a unique PDNF representation.

Defining a Map

Consider the space $\mathrm{X} = \prod_{P \in \mathcal{P}} \{0, 1\}$ with the product topology, where $\{0, 1\}$ has the discrete topology and consider the map $f : \mathcal{V} \to \mathrm{X}$ defined by $f(v) = \mathrm{x}$ where,

$$\pi_P(x) = \begin{cases} 1 & v(\mathrm{P}) = \textit{True} \\ 0 & v(\mathrm{P}) = \textit{False} \end{cases}$$

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if π_P is the P^{th} projection map.

Note

By the very definition, it is easy to see that f is a bijection.

f is continuous

We know that f is continuous iff $\pi_P \circ f$ is continuous $\forall$P $\in \mathcal{P}$. So we take P $\in \mathcal{P}$, and U to be any open set in $\{0, 1\}$ with the discrete topology.

Case: U $= \phi$: $(\pi_P \circ f)^{-1}$(U) $= \phi$ and hence, is open in $\mathcal{V}$

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Case: U $= \{1\}$: $(\pi_P \circ f)^{-1}(\text{U}) = \{v \in \mathcal{V}$ such that $v(\text{P}) = \mathit{True}\} = \text{X}_\alpha$ where $\alpha =$ P, and hence, is open in $\mathcal{V}$

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Case: U $= \{0\}$: $(\pi_P \circ f)^{-1}(\mathrm{U}) = \{v \in \mathcal{V}$ such that $v(\mathrm{P}) = \mathit{False}\} = \mathrm{X}_\alpha$ where $\alpha = \neg$P, and hence, is open in $\mathcal{V}$

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If α is a contradiction, $X_\alpha = \emptyset$ and hence, $f(X_\alpha) = \emptyset$ which is open.

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If α is a contradiction, $X_\alpha = \emptyset$ and hence, $f(X_\alpha) = \emptyset$ which is open.
Otherwise, α being a formula, $\exists\ \mathcal{P}' = \{P_1, P_2, \ldots.. P_n\} \subseteq \mathcal{P}$, such that α is made up of only elements from $\mathcal{P}'$, and hence can be converted into a PDNF
So, we can write α as a finite disjunction of clauses, where each clause is a finite conjunction of propositional constants, and hence α will be true if and only if any one of the clauses is true.

f is an open map

Let $\alpha \in \mathcal{W}$.
If α is a contradiction, $X_\alpha = \emptyset$ and hence, $f(X_\alpha) = \emptyset$ which is open.
Otherwise, α being a formula, $\exists\ \mathcal{P}' = \{P_1, P_2, \ldots.\ P_n\} \subseteq \mathcal{P}$, such that α is made up of only elements from $\mathcal{P}'$, and hence can be converted into a PDNF
So, we can write α as a finite disjunction of clauses, where each clause is a finite conjunction of propositional constants, and hence α will be true if and only if any one of the clauses is true.
Let there be k clauses.
Define for i $\in \{1, 2, \ldots.\ k\}$, and j $\in \{1, 2, \ldots\ n\}$,

$$b_{ij} = \begin{cases} 1 & P_j \text{ appears un-negated in the } i^{th} \text{ clause} \\ 0 & P_j \text{ appears negated in the } i^{th} \text{ clause} \end{cases}$$

f is an open map (contd...)

Now, $v \in \mathrm{X}_\alpha$.
$\Rightarrow$ v makes α true
$\Rightarrow$ v makes atleast one of the clauses in α true
$\Rightarrow$ $\exists$ i $\in$ {1, 2,....k} such that, v(P) =

$$\begin{cases} \textit{True} \text{ if } \mathrm{P} = \mathrm{P}_j \text{ for some } \mathrm{P}_j \in \{\mathrm{P}_1, \ldots.. \mathrm{P}_n\} \text{ and } \mathrm{b}_{ij} = 1 \\ \textit{False} \text{ if } \mathrm{P} = \mathrm{P}_j \text{ for some } \mathrm{P}_j \in \{\mathrm{P}_1, \ldots.. \mathrm{P}_n\} \text{ and } \mathrm{b}_{ij} = 0 \\ \textit{True} \text{ or } \textit{False} \text{ otherwise} \end{cases}$$

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Then,

$$f(\mathrm{X}_\alpha) = f\{v \in \mathcal{V} \text{ such that } v \text{ makes } \alpha \text{ true}\}.$$

$$= \bigcup_{i=1}^{k} \left(\bigcap_{j=1}^{n} \{\pi_{P_j}^{-1}(\{b_{ij}\})\} \right)$$

f is an open map (contd...)

We note that

$$\left(\bigcap_{j=1}^{n} \{ \pi_{P_j}^{-1}(\{b_{ij}\}) \} \right)$$

forms a basis for X with the product topology $\forall$i $\in \{1, 2, ...k\}$

$\Rightarrow f(X_\alpha)$ is a finite union of basic open sets

$\Rightarrow f(X_\alpha)$ open

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$\mathcal{V}$ and X are homeomorphic

So, we see that f is a homeomorphism and hence, $\mathcal{V}$ is homeomorphic to X

Defn: Cover, Sub-cover and Open Cover

Let $(X,\mathcal{T})$ be a topological space.

- If $\mathcal{C} = \{U_i\}_{i \in I}$ is an indexed family of sets U_i, then $\mathcal{C}$ is a cover of X if $X \subseteq \bigcup_{i \in I}\{U_i\}$.
- A subcover of $\mathcal{C}$ is a subset of $\mathcal{C}$ that still covers X.
- If $U_i \in \mathcal{T}$ $\forall i \in I$, then $\mathcal{C} = \{U_i\}_{i \in I}$ is called an open cover for X.

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Defn: Compact Spaces

Let $(X, \mathcal{T})$ be a topological space X. X is said to be compact if each of its open covers has a finite sub-cover.

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Result: Compactness is preserved under Homeomorphism

Let $(X, \mathcal{T})$, $(Y, \mathcal{T}')$ be topological spaces,
and $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a homeomorphism.
$A \subseteq X$ is compact in $(X, \mathcal{T})$ iff $f(A)$ is compact in $(Y, \mathcal{T}')$.

Result: The Tychonoff Theorem

Given an arbitrary family $\{X_\alpha\}_{\alpha \in \Lambda}$ of compact spaces, their product $X := \prod_{\alpha \in \Lambda} X_\alpha$ is compact.

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The Topological Space we defined is Compact

As f is a homeomorphism, $\mathcal{V}$ is compact iff X is.
As $\{0, 1\}$ with the discrete topology has finitely many open sets, it is compact.
$\Rightarrow$ By Tychonoff Theorem X is compact, and hence $\mathcal{V}$ is compact.

Defn: Finite Intersection Property

Let X be a set, and let $\mathcal{A} = \{A_i\}_{i \in I}$ be a family of subsets of X. Then, the collection $\mathcal{A}$ is said to have the finite intersection property (FIP), if any finite sub-collection $J \subseteq I$ has non-empty intersection (i.e. $\bigcap_{i \in J} A_i \neq \emptyset \; \forall$ finite $J \subseteq I$).

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Equivalent Definition of Compactness

Let $(X, \mathcal{T})$ be a Topological Space. X is compact if and only if any collection of closed subsets of X with the finite intersection property has non empty intersection.

X_α is closed $\forall \alpha \in \mathcal{W}$

We note that as f defined above is a homeomorphism, f, f^{-1} are well-defined continuous maps and hence, X_α is closed iff $f(X_\alpha)$ is closed.

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Now,

$$f(X_\alpha) = \bigcup_{i=1}^{k} \left(\bigcap_{j=1}^{n} \{\pi_{P_j}^{-1}(\{b_{ij}\})\} \right)$$

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Now,

$$f(X_\alpha) = \bigcup_{i=1}^{k} \left(\bigcap_{j=1}^{n} \{\pi_{P_j}^{-1}(\{b_{ij}\})\} \right)$$

As $\{b_{ij}\}$ is closed $\forall$i, j and π_{P_j} is continuous $\forall P_j \in \mathcal{P}'$, $\pi_{P_j}^{-1}(\{b_{ij}\})$ is closed in X with the product topology.

$\Rightarrow \bigcap_{j=1}^{n} \{\pi_{P_j}^{-1}(\{b_{ij}\})\}$ is closed in X with the product topology.

$\Rightarrow \bigcup_{i=1}^{k} \left(\bigcap_{j=1}^{n} \{\pi_{P_j}^{-1}(\{b_{ij}\})\} \right)$ is closed in X with the product topology.

$\Rightarrow f(X_\alpha)$ is closed in X with the product topology.

$\Rightarrow X_\alpha$ is closed in $\mathcal{V}$ with the topology defined above.

The Compactness Theorem of Propositinal Logic

Let $\mathcal{P}$ be a set of propositional constants, and let $\mathcal{W}$ be an infinite set of formulas over $\mathcal{P}$ such that every finite subset is satisfiable.

For each $\phi \in \mathcal{W}$, define $\mathsf{C}_\phi = \mathsf{X}_\phi$ which is a closed set.

Consider the collection of closed sets $\mathcal{C} = \{\mathsf{C}_\phi : \phi \in \mathcal{W}\}$.

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$\mathcal{C}$ has the finite intersection property

Consider any finite subcollection $\{\mathrm{C}_{\phi_1},...., \mathrm{C}_{\phi_n}\}$.

By assumption, since $\{\phi_1,...., \phi_1\}$ is a finite subset of $\mathcal{W}$, there is a truth assignment v that satisfies each of these ϕ_i.

Then $v \in \bigcap_{i=1}^{n} \mathrm{C}_{\phi_i}$, and so the intersection of any finite subcolleciton of $\mathcal{C}$ is nonempty.

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As $\mathcal{C}$ is compact, $\exists\ v \in \bigcap_{C \in \mathcal{C}} \mathrm{C}_{\phi_i}$.
By definition, this v satisfies ϕ, $\forall \phi \in \mathcal{W}$, and hence $\mathcal{W}$ is satisfiable.

The Compactness Theorem for Propositional Logic

Thus we have proved the Compactness Theorem for Propositional Logic which states:

Let $\mathcal{P}$ be a set of propositional constants, and $\mathcal{W}$ be an infinite set of formulas over $\mathcal{P}$. If every finite subset of $\mathcal{W}$ is satisfiable, so is $\mathcal{W}$.

Thankyou