Linear Matroid Intersection is in quasi-NC

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Matroids

Definition

$M = (E, \mathcal{I})$ is said to be a *matroid* if

E : finite set

$\mathcal{I}$: set of Independent sets satisfying:

1. Closure under subsets
2. Augmentation Property

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Some Notations:

1. $m = |E|$
2. $\mathcal{B} \subseteq \mathcal{I}$: collection of maximal sets in $\mathcal{I}$.
3. $\text{rank}(A) = \max\{|I| : I \in \mathcal{I} \text{ and } I \subseteq A\}$ for every $A \subseteq E$.

Linear Matroids and the Matroid Intersection Problem

Linear Matroids

For matrix $V = [v_1, v_2, \ldots, v_m]$, define $M_V = (E, \mathcal{I})$ as:

1. $E : \{1, 2, \ldots, m\}$.
2. $I \in \mathcal{I}$ iff $\{v_i : i \in I\}$ is a set of independent columns in V.

M is said to be a *linear matroid* if $M = M_V$ for some matrix V.

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Linear Matroid Intersection: Given M_1, M_2 with base sets $\mathcal{B}_1$, $\mathcal{B}_2$ respectively, find $B \in \mathcal{B}_1 \cap \mathcal{B}_2$ if it exists.

Linear Matroid Intersection is in RNC [NSV94]

Lemma

*U, V: $n \times m$ matrices; $\mathcal{B}_1$, $\mathcal{B}_2$: Bases of M_U, M_V.
Define Z to be an $m \times m$ diagonal matrix with diagonal entries $\{z_e\}_{e \in [m]}$ where z_e are variables and $D = UZV^T$. Then,*

$$M_U \text{ and } M_V \text{ have a common base iff } \det(D) \neq 0.$$

Proof. By Binet-Cauchy formula,

$$\det(D) = \sum_{B \subseteq [m],\ |B|=n} \left(\prod_{e \in B} z_e \right) \det(U_B) \det(V_B)$$

$$= \sum_{B \in \mathcal{B}_1 \cap \mathcal{B}_2} \left(\prod_{e \in B} z_e \right) \det(U_B) \det(V_B)$$

The Weight Function

$w : E \to \mathbb{Z}$ is a weight function.

Isolating Weight Function

w is said to be isolating for $\mathcal{A} \subseteq \mathcal{P}(E)$ if there is a unique minimum weight set in $\mathcal{A}$

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The Isolation Lemma [MVV87]

A random weight function with polynomially bounded weights is isolating for any family $\mathcal{A}$ with high probability.

The Plan and why it Works

1. Replace z_e in $\det(D)$ by $z^{w(e)}$ where z is a new variable. $\det(D)$ thus becomes a univariate in z and $\prod_{e \in B} z^{w(e)} = z^{w(B)}$.

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2. If w is an isolating weight assignment for $\mathcal{B}_1 \cap \mathcal{B}_2$, and $\mathcal{B}_1 \cap \mathcal{B}_2 \neq \emptyset$, then the minimum degree term in $\det(D)$ is unique. Thus, we get $\det(D) \neq 0 \Leftrightarrow \mathcal{B}_1 \cap \mathcal{B}_2 \neq \emptyset$.

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3. A determinant with small degree univariate entries can be computed in NC [BCP83].

Algorithm 1: Linear Matroid Intersection - RNC

Input : $n \times m$ arrays U, V - the arrays underlying the matroids M_1 and M_2 with bases $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively

Output: 0 if their bases have no intersection and a common base element B otherwise

```
Randomly choose a weight function w : E → [2m]
Let D be as described (with det(D) being a univariate in z)
if det(D) = 0 then
    return 0
B = ∅
for each e ∈ E do
    delete e and recompute det(D)
    if the minimum degree term disappears then
        B = B ∪ {e}
return B
```

Derandomising the RNC algorithm

What do we want?

Construct the weight assignment deterministically such that there is a unique *minimum weight* common base.

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How do they get such a sufficient condition?

Look at the Matroid as a Polytope to get a better handle.

The Matroid Polytope and Extending the Weight function

$M = (E, \mathcal{I})$ is a matroid.
For any $S \in \mathcal{P}(E)$, $x^S \in \mathbb{R}^{|E|}$ is its characteristic vector if

$$x_e^S = \begin{cases} 1 & \text{if } e \in S \\ 0 & \text{otherwise.} \end{cases}$$

The *matroid polytope* is $P(\mathcal{I}) = \text{conv}\left\{x^I : I \in \mathcal{I}\right\} \subseteq \mathbb{R}^{|E|}$.

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For any $x \in \mathbb{R}^{|E|}$ and $S \subseteq E$, define

$$w(x) = \sum_{e \in E} w(e) x_e,$$

$$w(S) = \sum_{e \in S} w(e).$$

Characterisation of the Matroid Polytope

For any $x \in \mathbb{R}^{|E|}$ and $S \subseteq E$, define

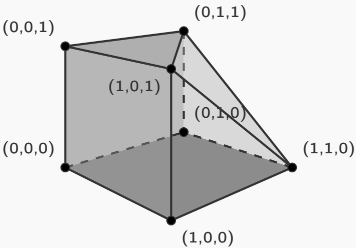
$$x(S) = \sum_{e \in S} x_e.$$

Lemma [Edm70]

For a matroid $M = (E, \mathcal{I})$, $x \in \mathbb{R}^{|E|}$ is in $P(\mathcal{I})$ iff

$$\forall e \in E, x_e \geq 0 \quad \text{and}$$

$$\forall S \subseteq E, x(S) \leq \text{rank}(S)$$



The underlying matrix:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The Matroid Base Polytope

Characterisation

$M = (E, \mathcal{I})$: matroid;
$\mathcal{B}$: family of base sets for M.
Then $P(\mathcal{B})$: Base Polytope.

The base polytope forms a face of the matroid polytope $P(\mathcal{I})$.

For any $x \in \mathbb{R}^{|E|}$, $x \in P(\mathcal{B})$ iff

$$\forall e \in E, x_e \geq 0,$$

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$$x(E) = \text{rank}(E)$$

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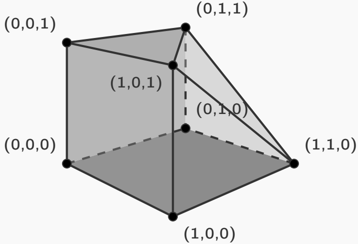
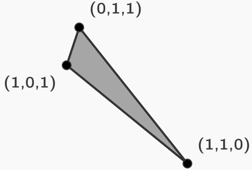


Its base polytope:



The faces are important

Observation

$w : E \to \mathbb{Z}$ is a weight assignment.
By definition, its extension to $\mathbb{R}^{|E|}$ is linear.
Thus the minimum weight points in the matroid base polytope form a face.

Notation: For $w : E \to \mathbb{Z}$, F_w is the face of min. weight points.

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The Plan ahead:

1. Find a condition to ensure that a face in $P(\mathcal{B}_1 \cap \mathcal{B}_2)$ contains just one point
2. Find a condition on w to ensure that F_w satisfies the above condition

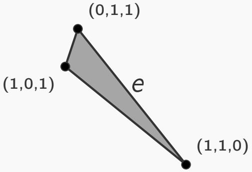
Faces of the Base Polytope

Characterisation

Any of the faces of the Matroid Base Polytope is described by additional equations of the form:

$$x_e = 0 \text{ or } x(S) = \text{rank}(S)$$

for some $S \subseteq E$.



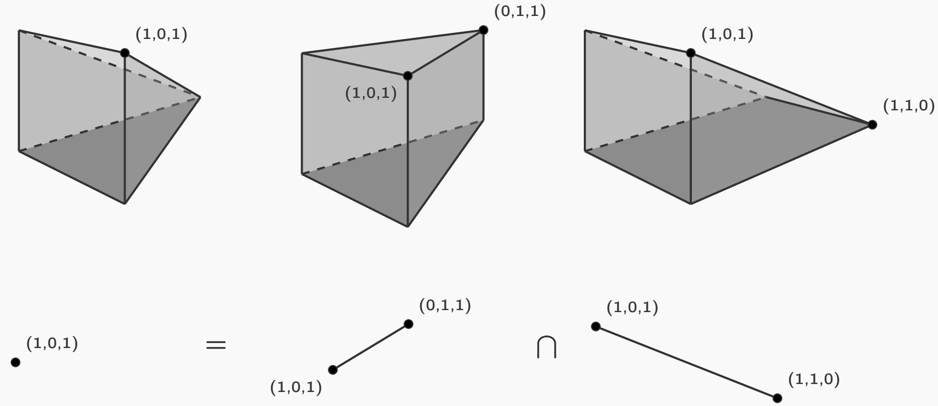
Faces of the Base Polytope:

1. $(1, 1, 0) : x_3 = 0$, $x(\{2\}) = \text{rank}(\{2\})$, $x(\{1, 2, 3\}) = \text{rank}(\{1, 2, 3\})$
2. $e : x(\{2\}) = \text{rank}(\{2\})$, $x(\{1, 2, 3\}) = \text{rank}(\{1, 2, 3\})$
3. The base polytope: $x(\{1, 2, 3\}) = \text{rank}(\{1, 2, 3\})$

Faces of the Base Intersection Polytope

Characterisation

Any face F of $P(\mathcal{B}_1 \cap \mathcal{B}_2)$ can be written as $F = F_1 \cap F_2$ for some faces F_1 and F_2 of $P(\mathcal{B}_1)$ and $P(\mathcal{B}_2)$ respectively.



Partitioning with respect to a face in the Intersection Polytope

$(E,\mathcal{I}_1)$, $(E,\mathcal{I}_2)$: matroids with base sets $\mathcal{B}_1$ and $\mathcal{B}_2$.
F: face of $P(\mathcal{B}_1 \cap \mathcal{B}_2)$.
Then, there exists partitions $\mathcal{S}$ and $\mathcal{T}$ of E satisfying:
$\forall S \in \mathcal{S},\ T \in \mathcal{T},\ \exists n_S, m_T \geq 0$ such that for any $x \in F$,

$$x(S) = n_S \text{ and } x(T) = m_T.$$

Moreover,

1. F satisfies $x(R) = \text{rank}_1(R)$ or $x(R) = \text{rank}_2(R)$
   $\Rightarrow R$ is a disjoint union of sets from $\mathcal{S}$ or $\mathcal{T}$ (resp.).
2. F satisfies $x_e = 0$
   $\Rightarrow \exists S \in \mathcal{S},\ T \in \mathcal{T}$ such that $S = T = \{e\}$ and $n_S = m_T = 0$.

Cycles in Matroid Intersection

F: a face of $P(\mathcal{B}_1 \cap \mathcal{B}_2)$ with partitions $\mathcal{S}$ and $\mathcal{T}$.
$C_F = (e_1, e_2, \ldots, e_{2r})$ is called a *cycle* with respect to the F, if:

For $i = \{1, 2, \ldots r\}$,

$$e_{2i-1}, e_{2i} \in S \quad \text{for some } S \in \mathcal{S},$$

$$e_{2i}, e_{2i+1} \in T \quad \text{for some } T \in \mathcal{T},$$

where $e_{2r+1} = e_1$.

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where $e_{2r+1} = e_1$.

Note:

Let F satisfy $x_e = 0$.
e can not appear in any cycle defined w.r.t. F because $\{e\}$ appears as a singleton in both the partitions for F.

Cycles always exist for faces with multiple bases

Lemma

Let B_1, B_2 be two bases in the face F of the polytope $P(\mathcal{B}_1 \cap \mathcal{B}_2)$. Then $B_1 \Delta B_2$ is a set of disjoint cycles.

Proof.

$x^{B_1} = (1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 1, 1, 1)$
$x^{B_2} = (0, 1, 1, 1, 0, 1, 1, 1, 1, 0, 0, 0, 1, 1)$

$\mathcal{S} = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8, 9, 10, 11, 12, 13, 14\}\} = \{S_1, S_2, S_3\}$
$\mathcal{T} = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8, 9, 10, 11, 12\}, \{13, 14\}\} = \{T_1, T_2, T_3\}$

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Proof.

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$$\mathcal{T} = \{\{1,2,3,4\}, \{5,6,7,8,9,10,11,12\}, \{13,14\}\} = \{T_1, T_2, T_3\}$$

1. Pick $12 \in B_1 \setminus B_2$ and let $C_1 = (12)$.
2. Note that $12 \in S_3$.
3. $|B_1 \cap S_3| = |B_2 \cap S_3|$ by property of $\mathcal{S}$.

4. Pick $9 \in (B_2 \cap S_3) \setminus (B_1 \cap S_3)$ and let $C_1 = (12, 9)$.
5. Note that $9 \in T_2$.
6. $|B_1 \cap T_2| = |B_2 \cap T_2|$ by property of $\mathcal{T}$.
7. Pick $12 \in (B_1 \cap T_2) \setminus (B_2 \cap T_2)$ and let $C_1 = (12, 9)$.
8. Set $B_1 = B_1 \setminus C_1$ and $B_2 = B_2 \setminus C_1$ and repeat.

The process stops when $B_1 = B_2$.

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Observations

Let $\mathcal{C}_F$ denote the family of all cycles with respect to the face F.

1. By the lemma: *If $\mathcal{C}_F = \emptyset$ then F has dimension 0.*
2. Going to a subface can only destroy cycles. That is:

For faces F, F' in $P(\mathcal{B}_1 \cap \mathcal{B}_2)$, $F' \subseteq F \Rightarrow \mathcal{C}_{F'} \subseteq \mathcal{C}_F$

Circulation of a Cycle

What we want: Successively eliminate cycles to reach smaller and smaller faces until we reach a face F with $\mathcal{C}_F = \varnothing$.

Circulations

For a weight assignment $w : E \to \mathbb{Z}$, the circulation $c_w(C)$ of a cycle $C = (e_1, e_2, \ldots, e_k)$ is defined as:

$$c_w(C) = |w(e_1) - w(e_2) + w(e_3) - \cdots - w(e_k)|$$

Why Circulations?

The cycles on the minimum weight face have a property which can be explained easily using it.

Cycles on the face the of Minimum Weight Common Bases

Lemma

For any face F in $P(\mathcal{B}_1 \cap \mathcal{B}_2)$, if the weight function w is such that $w(x)$ is constant on F, then $c_w(C) = 0$ for any $C \in \mathcal{C}_F$

Proof.

$C = (15, 3, 7, 11, 19, 1, 20, 18, 12, 8, 5, 10)$

$C_1 = (15, 7, 19, 20, 12, 5)$, $C_2 = (3, 11, 1, 18, 8, 10)$.

Circulation vector: $\delta_C = x^{C_1} - x^{C_2}$

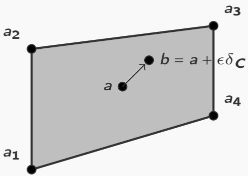
$$= (-1, 0, -1, 0, 1, 0, 1, -1, 0, -1, -1, 1, 0, 0, 1, 0, 0, -1, 1, 1)$$

$w = (1, 7, 4, 8, 3, 9, 4, 2, 10, 5, 6, 8, 2, 9, 6, 10, 1, 4, 7, 3)$

$c_w(C) = w.\delta_C = -1 - 4 + 3 + 4 - 2 - 5 - 6 + 8 + 6 - 4 + 7 + 3$

Proof (contd.)

We show that in the current setting $w.\delta_C = 0$.



$$a = \frac{a_1 + a_2 + a_3 + a_4}{4}$$

$$S_0 = \{e \in E \mid x_e = 0 \forall x \in F\}$$

$$\mathcal{R}_1 = \{R \subseteq E \mid x(R) = \text{rank}_1(R) \forall x \in F\}$$

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Case 1: For $e \in S_0$, $e \notin C$. Thus, $b_e = a_e = 0$.

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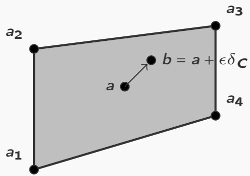

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Case 2: Let $R \in \mathcal{R}_1$ and $\mathcal{S}$: Partition of E.

$|C_1 \cap S| = |C_2 \cap S|$ for any $S \in \mathcal{S}$

$\Rightarrow \delta_C(S) = 0$ for every $S \in \mathcal{S}$.

R is a disjoint union of sets from $\mathcal{S}$

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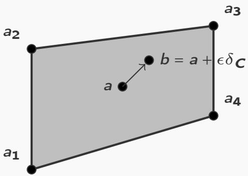

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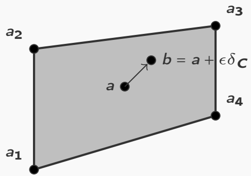

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Case 3: Similar (Take $R \in \mathcal{R}_2$ and $\mathcal{T}$: Partition of E)

$$a + \epsilon\delta_C \in F \Rightarrow w(a) = w(a + \epsilon\delta_C) \Rightarrow w(\delta_C) = w.\delta_C = 0.$$

The Picture gets a little Clearer...

Putting it all together...

w: a weight function;
C: a cycle on some face of $P(\mathcal{B}_1 \cap \mathcal{B}_2)$.
Then,

$$c_w(C) \neq 0 \Rightarrow C \notin \mathcal{C}_{F_w}.$$

Thus, a sufficient condition for w to be isolating:

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What we need to do: Ensure that such a w can be found.

Isolating Weight Assignments can be found

Lemma [CRS95]

For any number s, a set of $O(m^2 s)$ integer weight functions on the set E can be constructed with weight bounded by $O(m^2 s)$ in time $\text{poly}(ms)$ such that for any set of s cycles, one of the weight functions will give non-zero ciclulation to each of the s cycles.

Proof. Let $E = \{e_1, e_2, \ldots, e_m\}$.
Define $w(e_i) = 2^{i-1}$ for each $i \in \{1, 2, \ldots, n\}$.
Consider $\mathcal{W} = \{w \pmod{j} \mid 2 \leq j \leq t\}$ for some t to be decided.
Let $\{C_1, C_2, \ldots C_s\}$ be a set of s cycles.

What we want:

$$\exists j \leq t \ \forall i \leq s, c_{w \pmod{j}}(C_i) \neq 0.$$

Proof (contd.)

However,

$$\exists j \leq t \ \forall i \leq s, c_{w \pmod{j}}(C_i) \neq 0 \Leftarrow \exists j \leq t, \prod_{i=1}^{s} c_w(C_i) \neq 0 \pmod{j}$$
$$\Leftarrow \operatorname{lcm}(2,3,\ldots,t) \not| \prod_{i=1}^{s} c_w(C_i)$$
$$\Leftarrow \operatorname{lcm}(2,3,\ldots,t) \geq \prod_{i=1}^{s} c_w(C_i).$$

Also,

$$\prod_{i=1}^{s} c_w(C_i) \leq 2^{m^2 s}.$$

and by [Nai82], $\operatorname{lcm}(2,3,\ldots,t) \geq 2^t$ for $t \geq 7$.

Thus, $t = m^2 s$ suffices.

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Observation

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The Plan Ahead:

1. Ensure that if cycles of a certain length are removed, then not too many cycles of double this length remain.
2. Eliminate cycles in rounds - *In each round, double the length of the cycles eliminated.*

Eliminating Cycles

Lemma

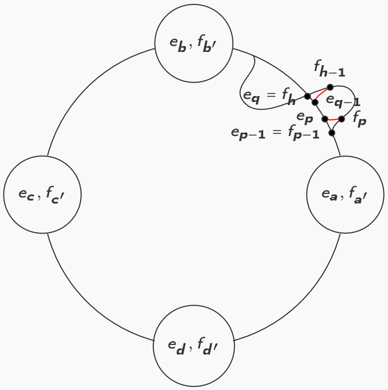
Let F be a face of $P(\mathcal{B}_1 \cap \mathcal{B}_2)$. If $\mathcal{C}_F$ has no cycles of length $\leq r$, for some even number number $r \geq 2$, then $\mathcal{C}_F$ has $\leq m^4$ cycles of length $\leq 2r$.

Proof. $\mathcal{S}$ and $\mathcal{T}$: Partitions of E w.r.t. F.
$C = (e_0, e_1, \ldots, e_{s-1})$ is a cycle of length $s \leq 2r$.
Choose fours points from C that divide it into four equal parts:

$$(a, b, c, d) = (0, \lceil s/4 \rceil, \lceil s/2 \rceil, \lceil 3s/4 \rceil)$$

Claim: The tuple (e_a, e_b, e_c, e_d) uniquley describes C.
There are at most m^4 ways of choosing the tuple.

Proof of the Claim



$C' = (f_0, f_1, \ldots, f_{t-1})$;
$(a', b', c', d') = (0, \lceil t/4 \rceil, \lceil t/2 \rceil, \lceil 3t/4 \rceil)$;
$(e_a, e_b, e_c, e_d) = (f_{a'}, f_{b'}, f_{c'}, f_{d'})$.

$e_{p-1} = f_{p-1} \Rightarrow e_p,\ f_p$ belong to a common $S \in \mathcal{S}$ or $T \in \mathcal{T}$.

Case 1: q and h have same parity.
$e_q = f_h \Rightarrow e_{q-1},\ f_{h-1}$ belong to a common $S \in \mathcal{S}$ or $T \in \mathcal{T}$.

Case 2: q and h have different parity.
$e_{q-1},\ f_h$ belong to a common $S \in \mathcal{S}$ or $T \in \mathcal{T}$.

In both cases, there is a cycle of length $\leq r$. Contradiction.

The Isolating Weight Construction

Given:

$(E, \mathcal{I}_1)$, $(E, \mathcal{I}_2)$: Matroids with collection of base sets $\mathcal{B}_1$, $\mathcal{B}_2$.

The Construction:

Let $m = |E|$ and $t = \lceil \log m \rceil$. Define $F_0 = P(\mathcal{B}_2 \cap \mathcal{B}_2)$.

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For $i = 0, 1, \ldots, t-1$, define:

- w_i = weight assignment such that $c_{w_i}(C) \neq 0$, for any cycle $C \in \mathcal{C}_{F_i}$ of length 2^{i+1}.
- F_{i+1} = set of minimum weight points of F_i with respect to w_i.

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Let N be greater than any of weights in $w_0, w_1, \ldots, w_{t-1}$.

For $i = 0, 1, \ldots, t-1$, define:

$$W_i = w_0 N^i + w_1 N^{i-1} + \cdots + w_i N^0.$$

The final weight assignment: W_{t-1}.

The quasi-NC Algorithm

For each $i = 0, 1, \dots, t-1$,

- w_i has $O(m^6)$ possibilities.
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Weights are quasi-polynomially bounded $\Rightarrow$ Entries of the determinant are expressible in quasi-polynomially many bits.

By [Ber84, BCP83], the determinant can be computed in quasi-NC.

Correctness

Claim

F_{i+1} is set of minimum weight points in $P(\mathcal{B}_1 \cap \mathcal{B}_2)$ w.r.t. W_i for $i = 0, 1, \ldots, t-1$.

Proof. By Induction.

Clearly true for $i = 0$. Assume true for F_i. Then,
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Now

$$N > w_i \Rightarrow NW_{i-1} > w_i \text{ and } W_i = NW_{i-1} + w_i.$$

Thus, $W_i(x)$ is minimum for $x \in P(\mathcal{B}_1 \cap \mathcal{B}_2)$ if and only if

$$NW_{i-1}(x) \text{ is minimum for } x \in P(\mathcal{B}_1 \cap \mathcal{B}_2) \text{ (that is, } x \in F_i)$$

and $w_i(x)$ is minimum.

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and $w_i(x)$ is minimum. Thus, $F_{i+1} \subseteq F_i$ and

$$W_i(x) \text{ is minimum for } x \in P(\mathcal{B}_1 \cap \mathcal{B}_2) \Leftrightarrow x \in F_{i+1}$$

Correctness (contd.)

Claim

For $i = 1, 2, \ldots, t$, $\mathcal{C}_{F_i}$ has no cycles of length $\leq 2^i$.

Proof. For $i = 1, 2, \ldots, t$,

$$c_{w_{i-1}}(C) \neq 0 \text{ for every } C \in \mathcal{C}_{F_{i-1}} \text{ of length } \leq 2^i$$

but w_{i-1} is constant over $F_i \Rightarrow c_{w_{i-1}}(C) = 0$ for every $C \in \mathcal{C}_{F_i}$.
As $C_{F_i} \subseteq C_{F_{i-1}}$, $\mathcal{C}_{F_i}$ has no cycles of length $\leq 2^i$.

Correctness (contd.)

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Lemma

W_{t-1} is isolating.

Proof. The face minimised by W_{t-1} is F_t. That is, $F_{W_t} = F_t$.
$\mathcal{C}_{F_t}$ has no cycles of length $\leq 2^t = m$. That is, $\mathcal{C}_{F_t} = \emptyset$.

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