A Quadratic lowerbound for Homogeneous ABPs

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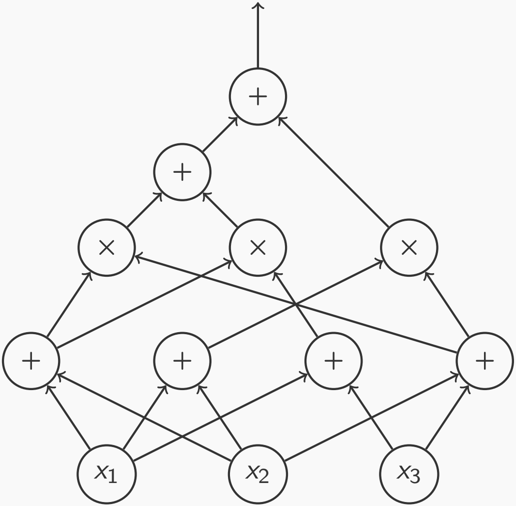
January 19, 2018

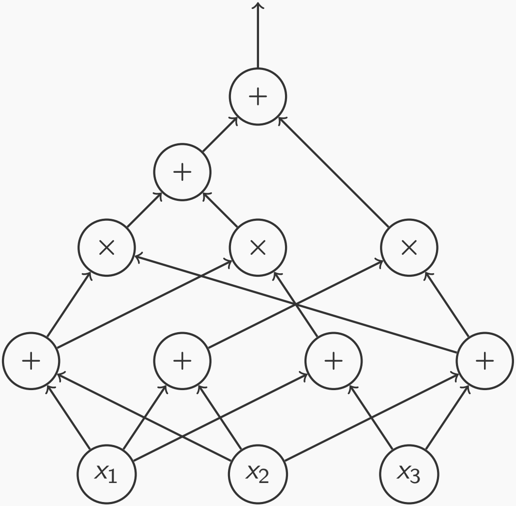
Algebraic Circuits



Fan-in 2

Size = Number of Gates

Algebraic Circuits



Homogeneous Circuits

Every gate computes a homogeneous polynomial

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Note: Lowerbound is on the number of wires and not on the number of gates.

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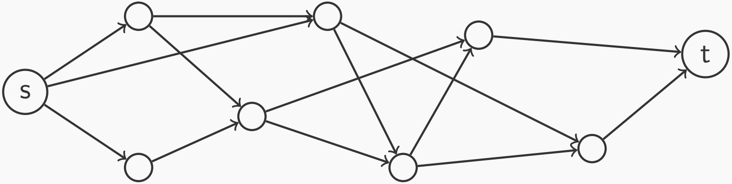
Formulas $\subseteq$ ABPs $\subseteq$ Circuits

Algebraic Branching Programs



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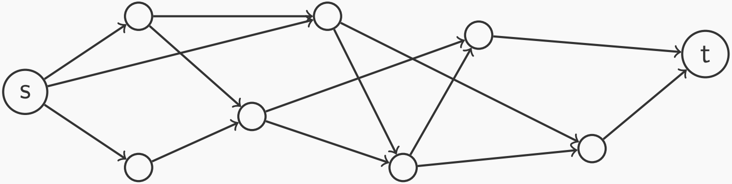

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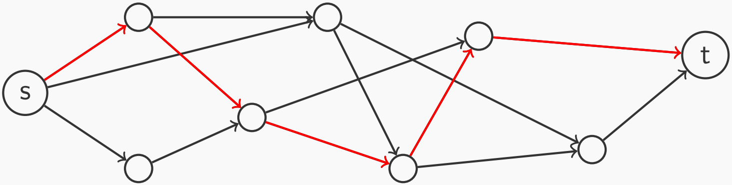

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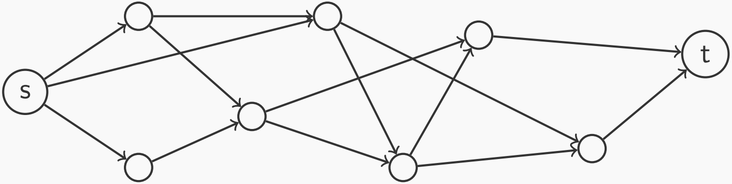

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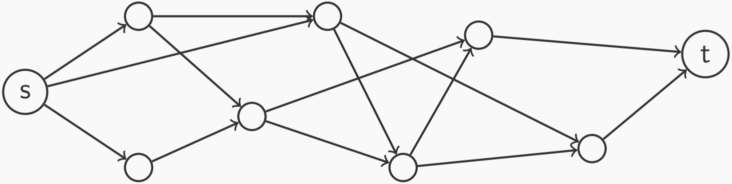

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Note: Best known lowerbound is still $\Omega(n \log n)$ [BS83]

The Main Result

Let $\mathbb{F}$ be an algebraically closed field of characteristic zero or relatively prime to d. Let B be any homogeneous ABP over the field $\mathbb{F}$ which computes the polynomial

$$P_{n,d} = \sum_{i=1}^{n} x_i^d.$$

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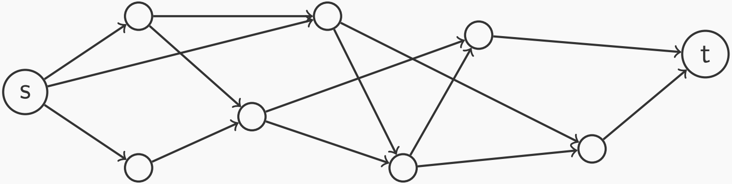
Proof Strategy



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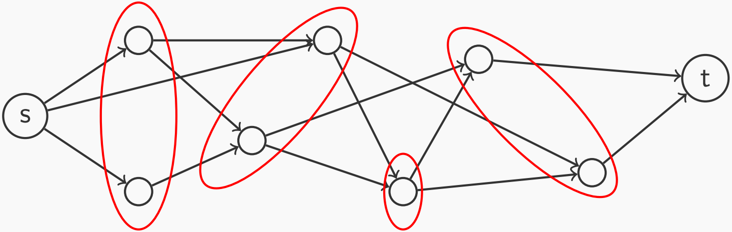

Step 0: Partition the vertex set.

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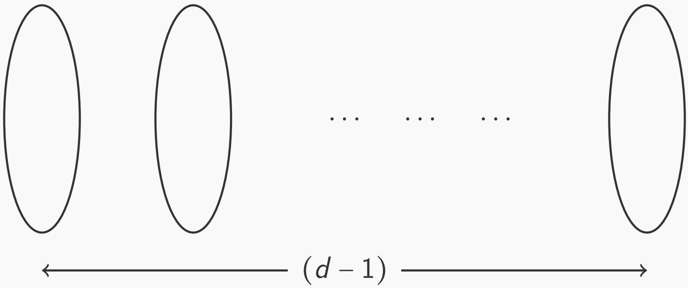

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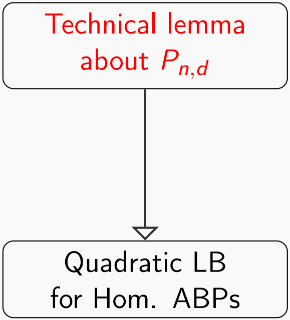
Step 2: Show that the size of each part is "large" if the ABP is computing $P_{n,d}$.

Contributions



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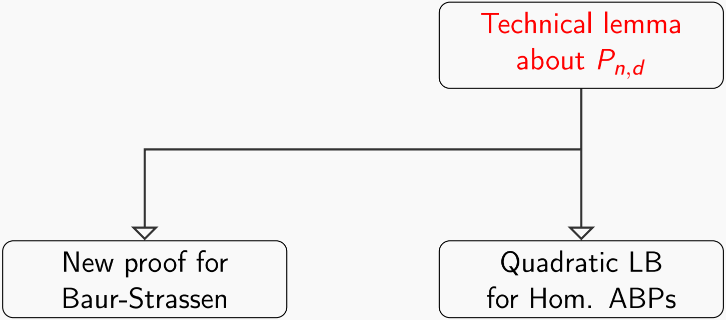

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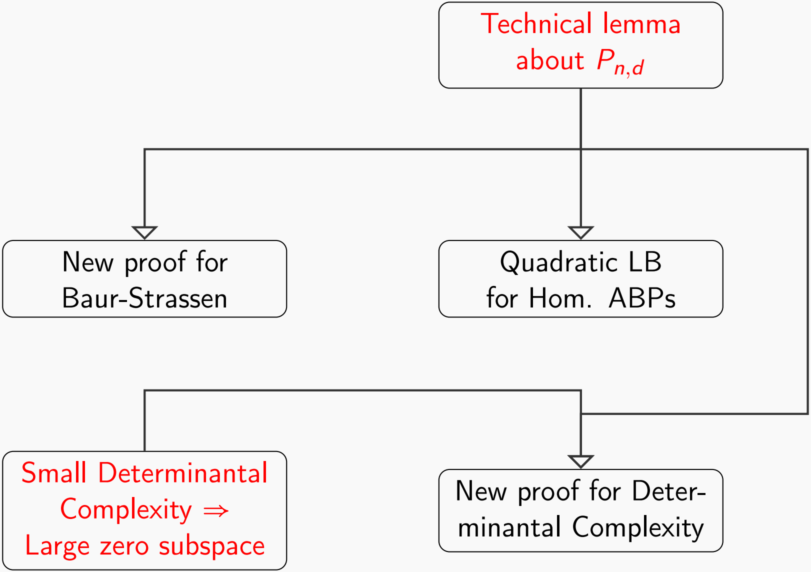

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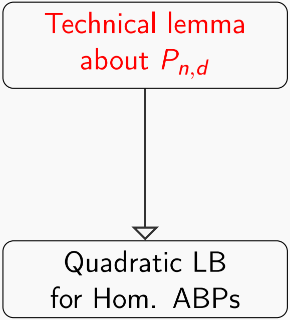

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From now on, assume that we are working over $\mathbb{C}$.

The Homogeneous Case

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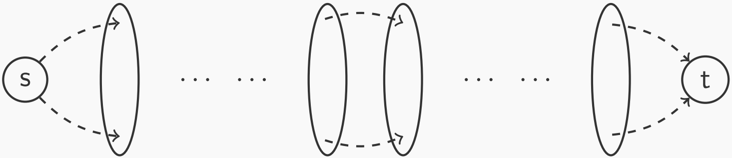
Get it in Shape



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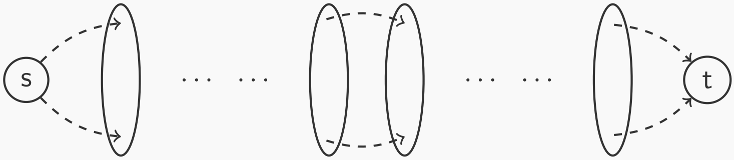

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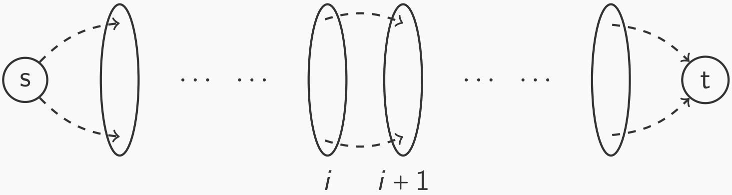

- S_i: Vertices computing polynomials of degree i

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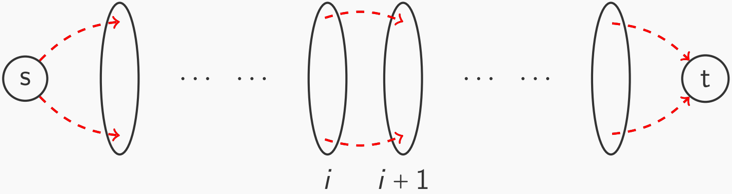

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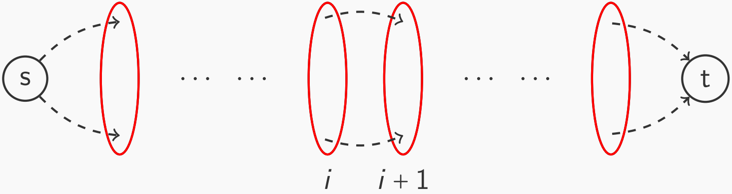

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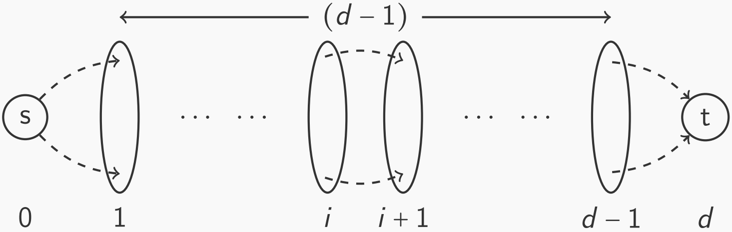

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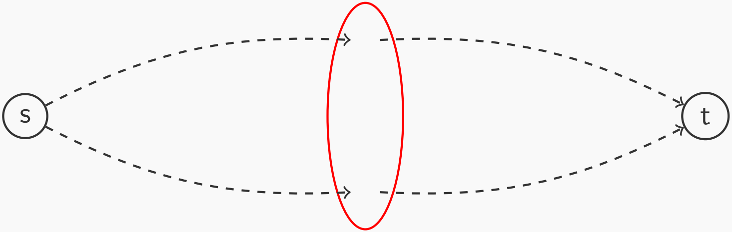

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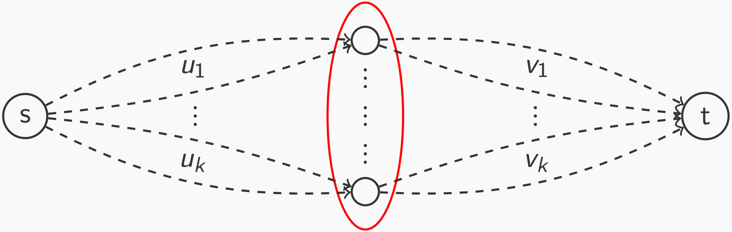

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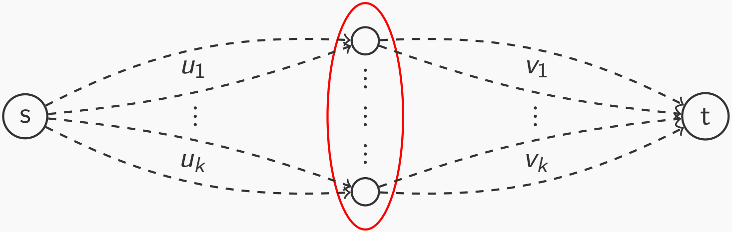

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Lemma [Kayal]: In the above case, $k \geq n/2$.

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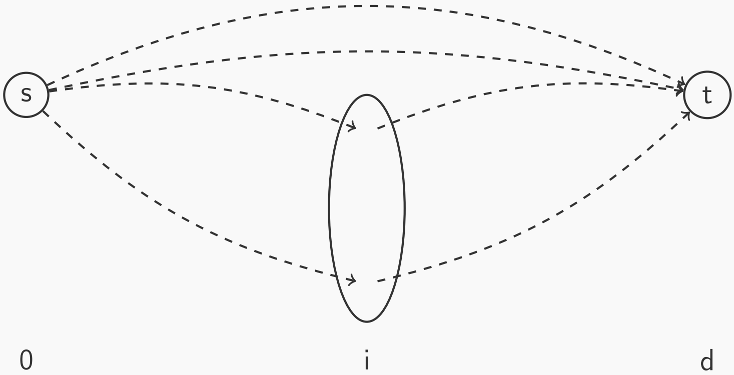

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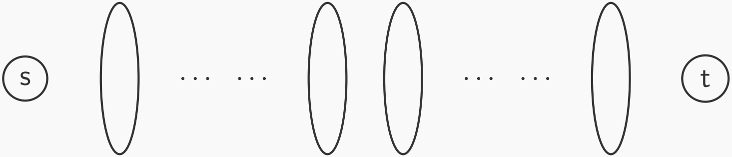

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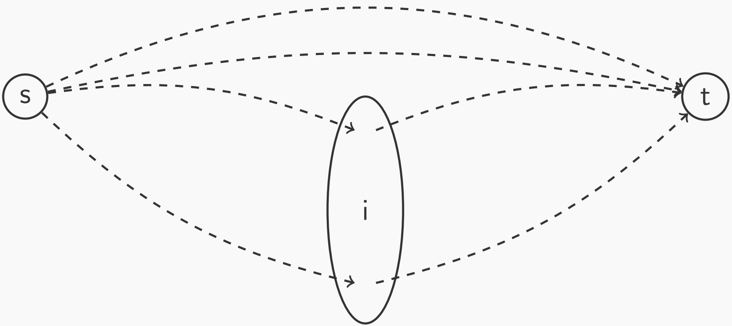

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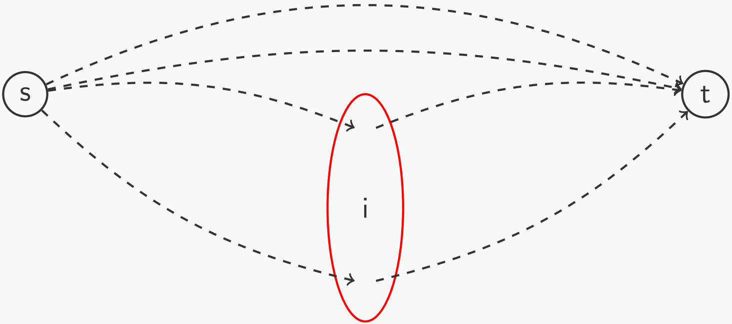

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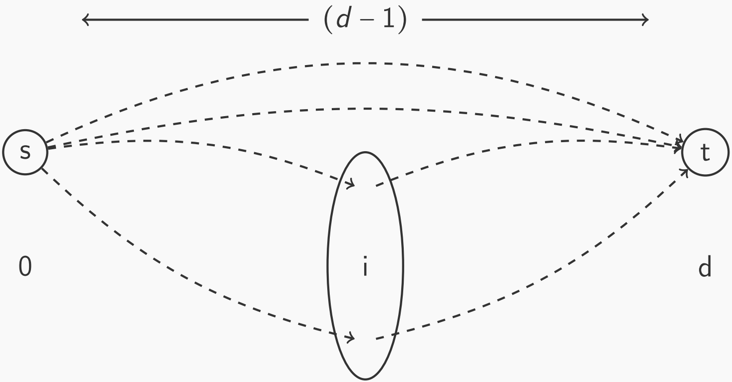

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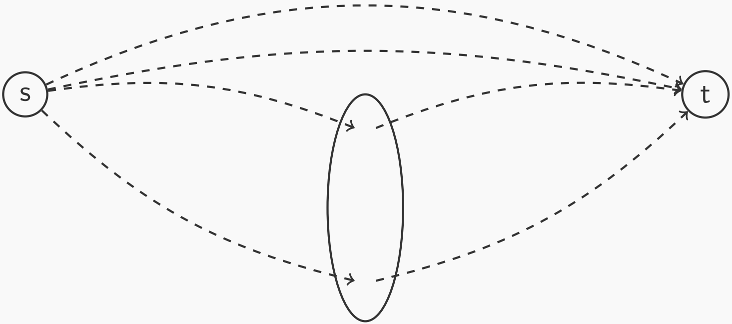

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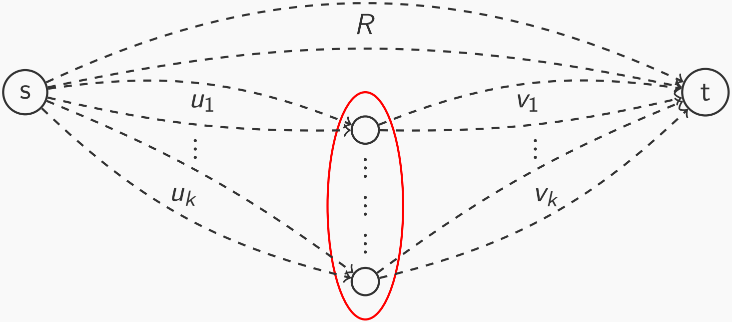

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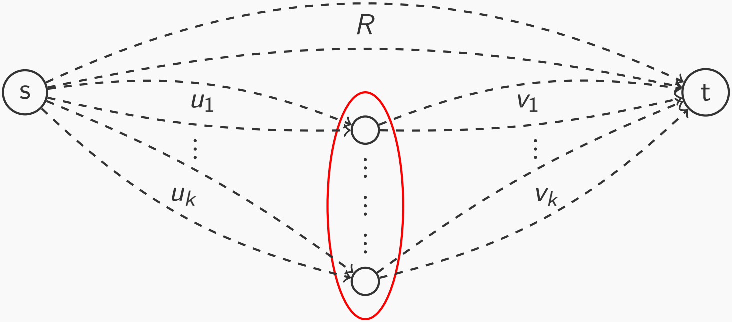

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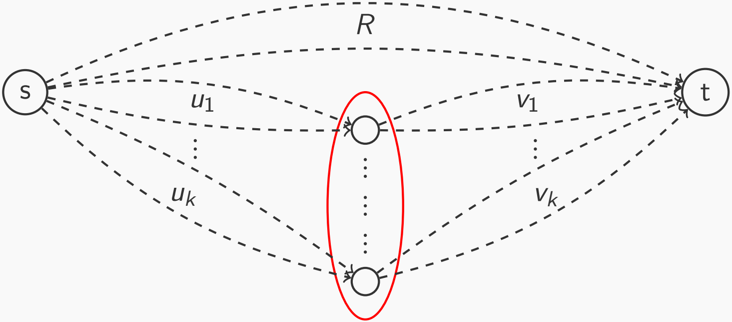

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Claim: In this case too, $k \ge n/2$.

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$$P_{n,d} = R' + \sum_{i=1}^{k} u_i' v_i'$$

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Lemma: For polynomials $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k$ with $1 \le \deg(u_i), \deg(v_i) \le d-1$ and R with $\deg(R) \le d-1$, if

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Proof: V - set of common zeros of $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$.
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Another Lemma

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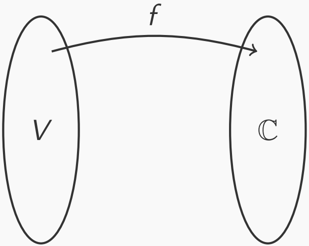

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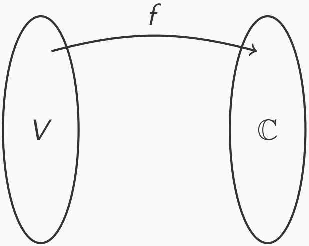

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$\forall i, x_i^d \equiv g_i$ for every $\mathbf{x} \in V$

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