Lower Bounds in Algebraic Circuit Complexity

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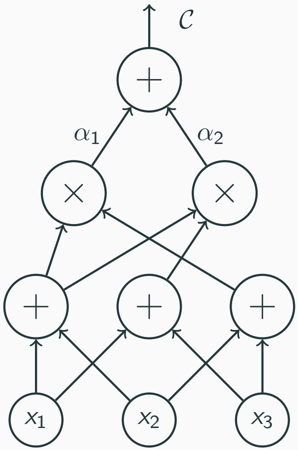
Can the given polynomial be computed efficiently?

Algebraic Models of Computation



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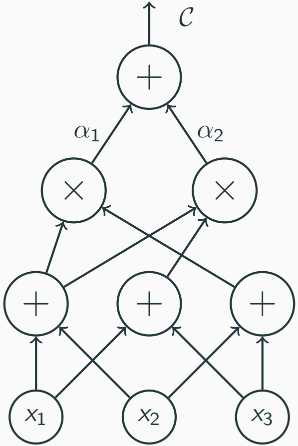
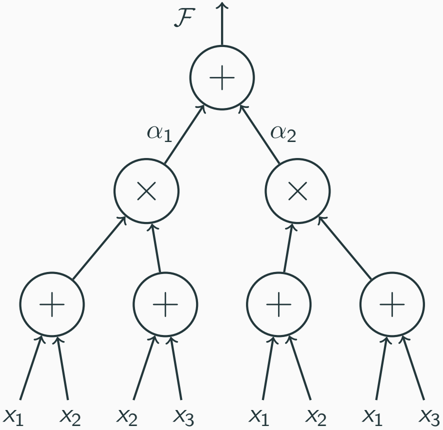
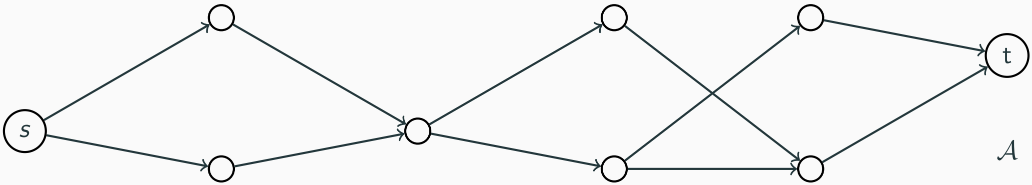




Algebraic Branching Programs



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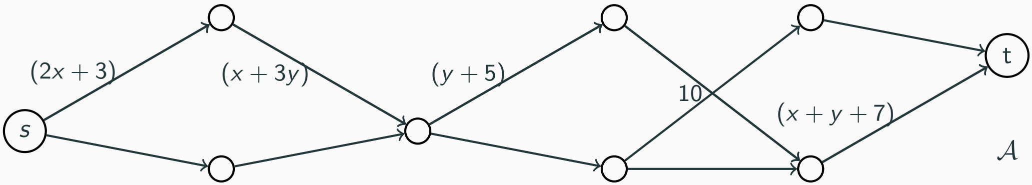

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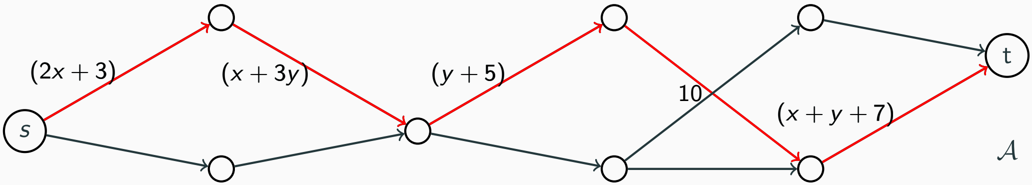

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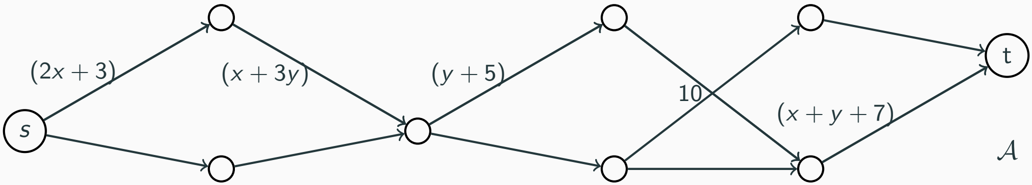

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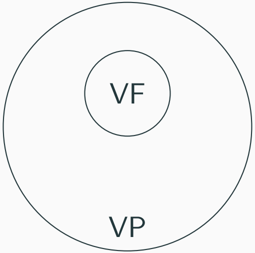

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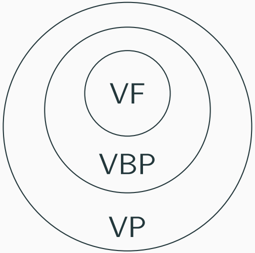

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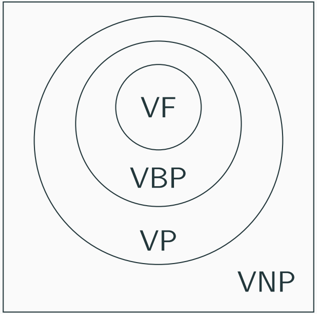

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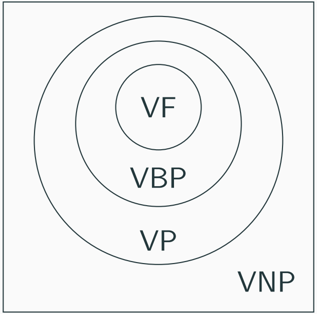

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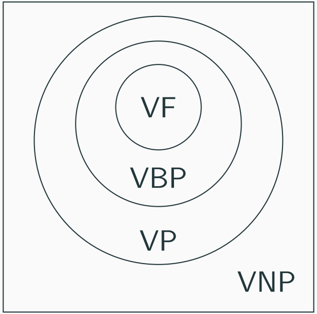

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Central Question: Find explicit polynomials that cannot be computed by efficient circuits.

Some Natural Restrictions

$$\boxed{\mathrm{Det}_n(\mathbf{x}) = \sum_{\sigma \in S_n} (-1)^{\mathsf{sgn}(\sigma)} \prod_{i=1}^{n} x_{i\sigma(i)}}$$

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Constant Depth Circuits

Length of the longest root-to-leaf path is a constant independent of n.

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The lower bound is $n^{\Omega(\sqrt{d})}$ for depth-3 and depth-4, proving that the depth reduction statements are tight.

Successes in Other Restricted Models

Multilinear Setting

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[Cha]: There is a tight separation between ABPs and some syntactically structured formulas.

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[C-Kumar-She-Volk]: Any formula computing $\mathrm{ESym}_{n,0.1n}(\mathbf{x})$ requires $\Omega(n^2)$ vertices, where

$$\mathrm{ESYM}_{n,d}(\mathbf{x}) = \sum_{i_1 < \cdots < i_d \in [n]} x_{i_1} \cdots x_{i_d}.$$

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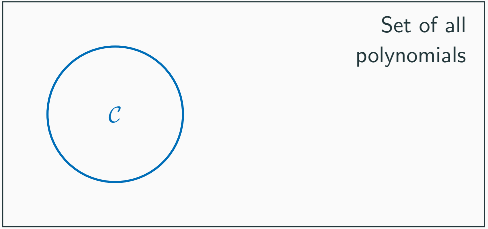

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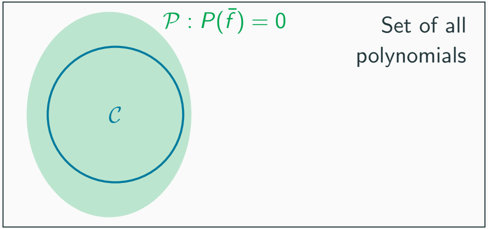

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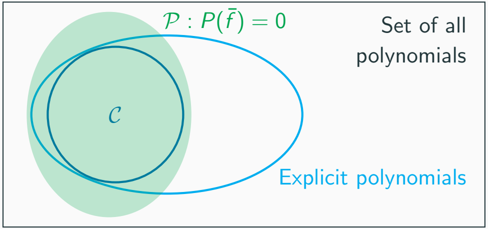

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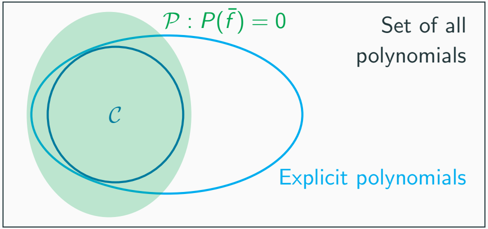

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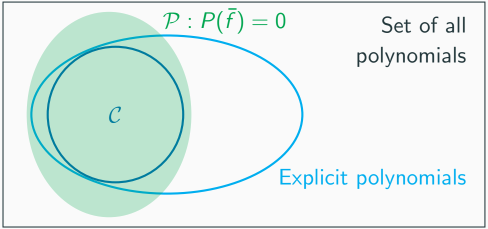

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[K-R-S-T]: Suppose the Permanent polynomial is $2^{n^{\varepsilon}}$-hard for constant $\varepsilon > 0$. In this case, if VP has natural proofs, then there is also a natural proof P that has explicit non-roots.

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- Does VP have natural proofs? Maybe under some natural assumptions?

Questions?

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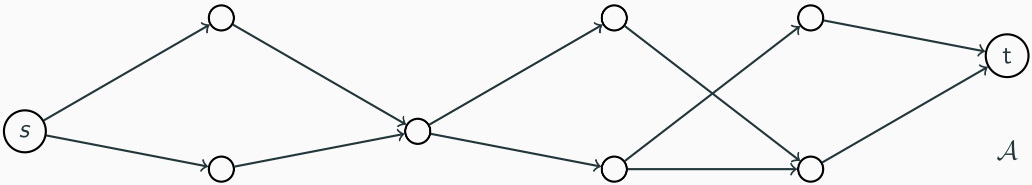

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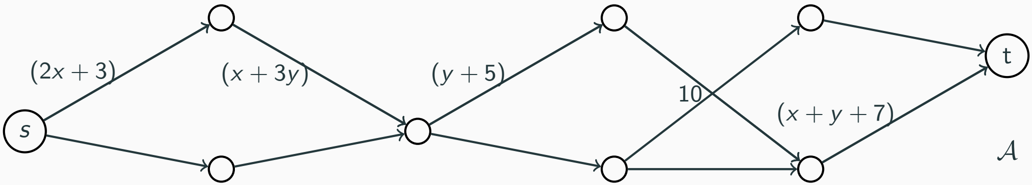

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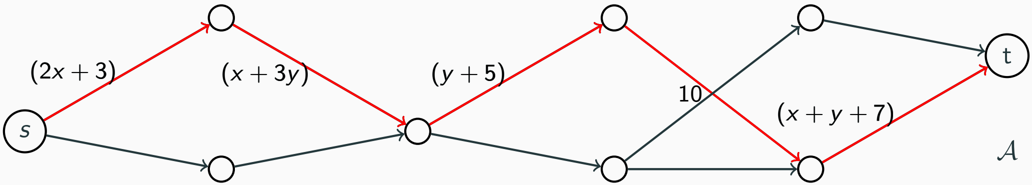

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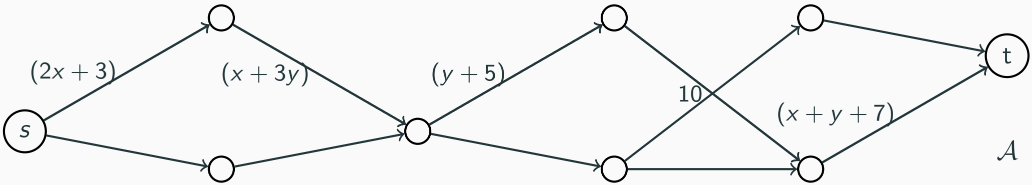

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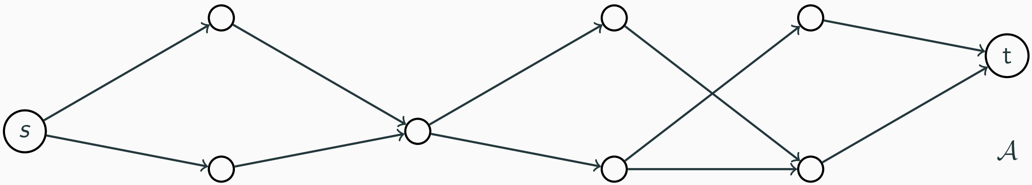

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Question: Is there an explicit polynomial that can not be computed by efficient ABPs?

Lower Bounds Against ABPs

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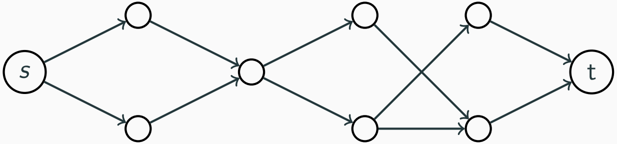
Our Result

[C-Kumar-She-Volk]: Any ABP computing $\sum_{i=1}^{n} x_i^d$ requires $\Omega(nd)$ vertices.

The Homogeneous Case: Proof Overview



$P_{n,d}(\mathbf{x}) = \sum_{i=1}^{n} x_i^d$.

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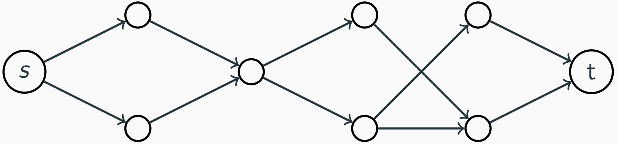

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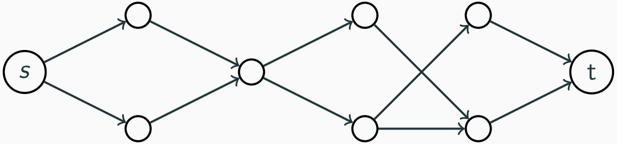

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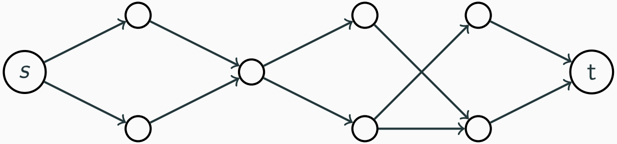

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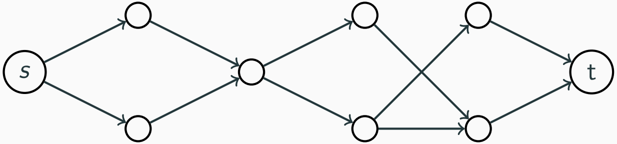

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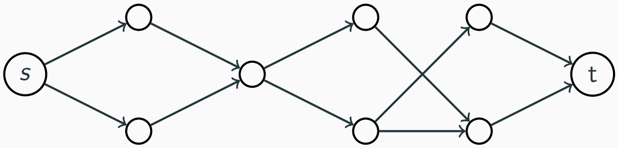

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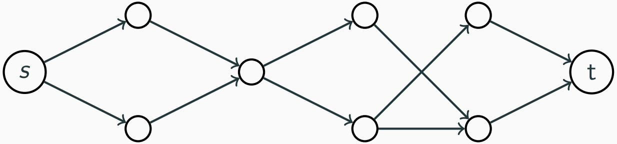

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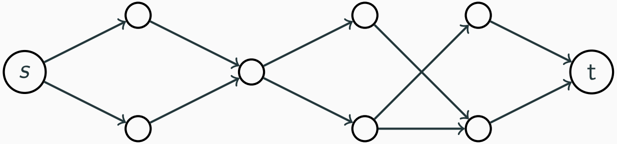

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Note: $\deg(g_i'), \deg(h_i') \leq [d-1]$ for every $i \in [d-1]$. Thus $\deg(R) \leq d - 1$.

The Homogeneous Case: Proof Overview (contd.)

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The Homogeneous Case: Proof Overview (contd.)

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Look at the first order derivatives.

$$S = \{\partial_{x_i}(P_{n,d} - R)\} = \left\{d \cdot x_i^{d-1} - \partial_{x_i}(R)\right\}_{i \in [n]}$$

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$$\mathcal{V}' \subseteq \mathcal{V}$$

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$R = 0$

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Proof Overview Of Our Result

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Any ABP with $(d+1)$ layers computing $\sum_{i=1}^{n} x_i^d$ has $\Omega(nd)$ vertices.

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where $A_i(0) = 0 = B_i(0)$ and $\deg(\delta(\mathbf{x})) < d$, has at least

$((n/2) - r) \cdot (d - 1)$ vertices.

Proof Overview Of Our Result (contd.)

Step 2: Iteratively reduce to Base Case

In each iteration, reduce the number of layers till it becomes $(d+1)$ such that

Proof Overview Of Our Result (contd.)

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In each iteration, reduce the number of layers till it becomes $(d+1)$ such that

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Proof Overview Of Our Result (contd.)

Step 2: Iteratively reduce to Base Case

In each iteration, reduce the number of layers till it becomes $(d+1)$ such that

- the number of layers is reduced by a constant fraction,
- the size does not increase,
- the polynomial being computed continues to look like

$$f_{\ell+1} = \sum_{i=1}^{n} x_i^d + \sum_{i=1}^{r_{\ell+1}} A_i(\mathbf{x}) \cdot B_i(\mathbf{x}) + \delta_{\ell+1}(\mathbf{x})$$

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where $\quad A_i(0) = 0 = B_i(0) \quad$ and $\quad \deg(\delta_{\ell+1}(\mathbf{x})) < d$,

- number of error terms collected is small.

The Induction Step

ℓ-th step

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Given: $\mathcal{A}_\ell$

Size $= s_\ell$
Number of layers $= d_\ell$
Number of error terms $= r_\ell$

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Want to construct: $\mathcal{A}_{\ell+1}$

Size $= s_{\ell+1}$

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Number of Steps: $\Theta(\log n)$

The Induction Step

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Size $= s_\ell$
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Size $= s_{\ell+1} \leq s_\ell$
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Number of error terms $= r_{\ell+1} \leq r_\ell + \frac{s_\ell}{d_\ell/3}$

Number of Steps: $\Theta(\log n)$

Number of Error Terms: $\varepsilon \cdot n$ where ε can be chosen.

The Induction Step

ℓ-th step

Given: $\mathcal{A}_\ell$

Size $= s_\ell$

Number of layers $= d_\ell$

Number of error terms $= r_\ell$

Want to construct: $\mathcal{A}_{\ell+1}$

Size $= s_{\ell+1} \leq s_\ell$

Number of layers $= d_{\ell+1} \leq \frac{2}{3} d_\ell$

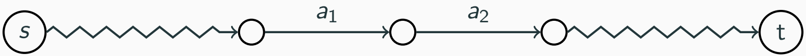
Number of error terms $= r_{\ell+1} \leq r_\ell + \frac{s_\ell}{d_\ell/3}$

Number of Steps: $\Theta(\log n)$

Number of Error Terms: $\varepsilon \cdot n$ where ε can be chosen.

The Lower Bound: $((n/2) - \varepsilon \cdot n) \cdot (d - 1)$

Proof of the Induction Step



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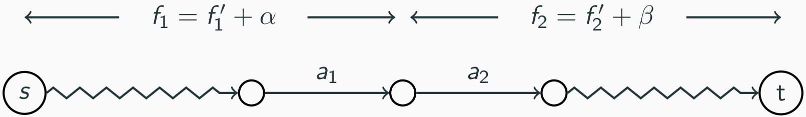
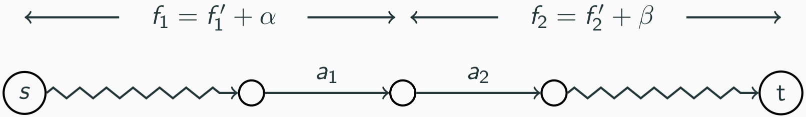


Proof of the Induction Step

$\mathcal{A}_\ell = f_1 \cdot f_2$



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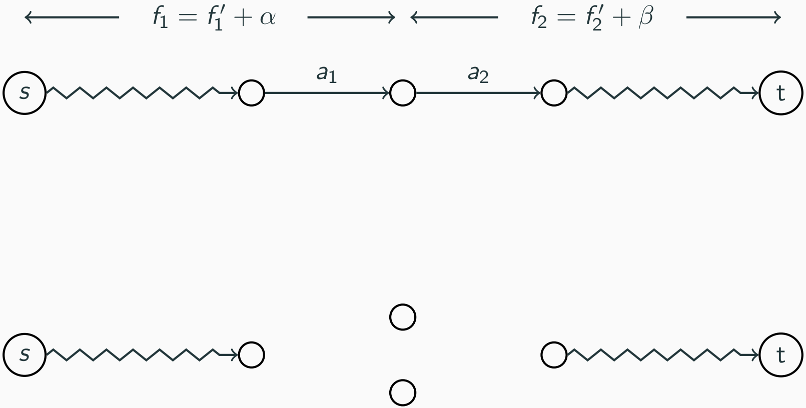

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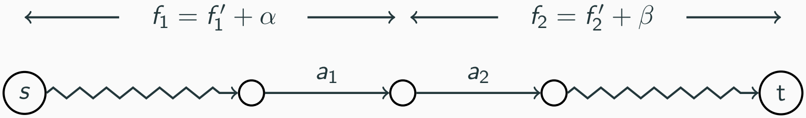

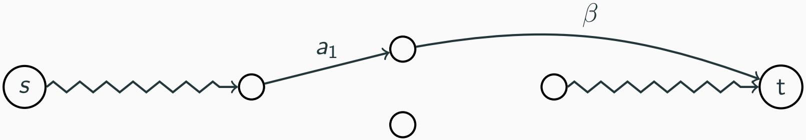

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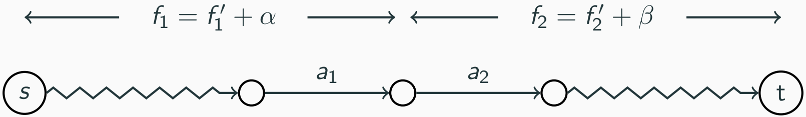

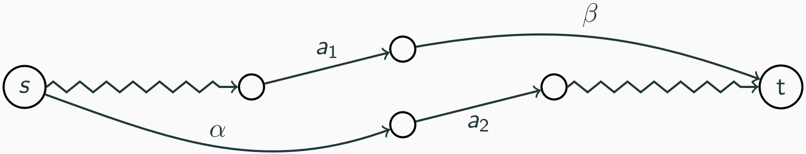

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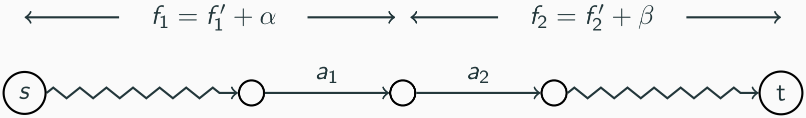
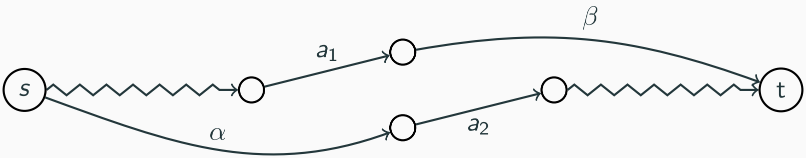


$\mathcal{A}_{\ell+1} = \beta \cdot f_1 + \alpha \cdot f_2$



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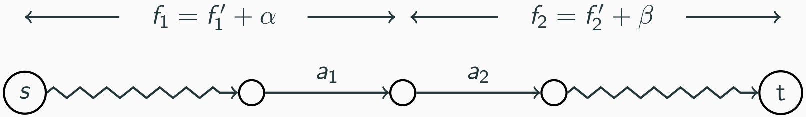

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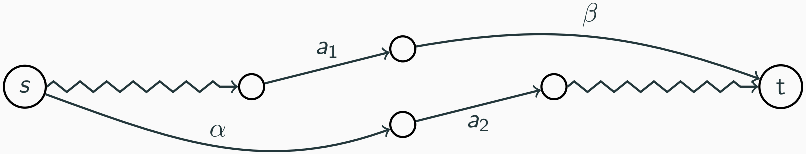

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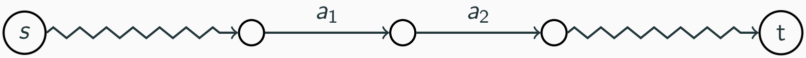

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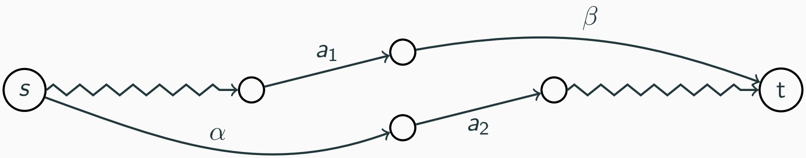

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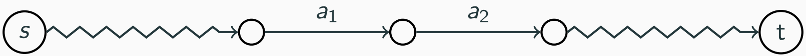

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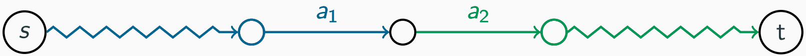

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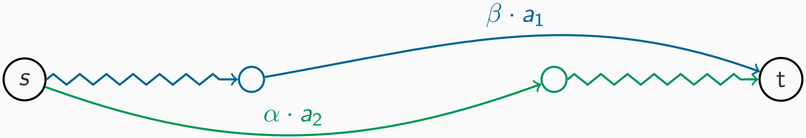

Algebraic Formulas And Lower Bounds Against Them

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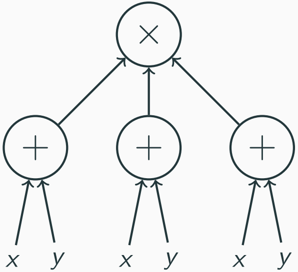
$$(x+y)\cdot(x+y)\cdot(x+y)$$



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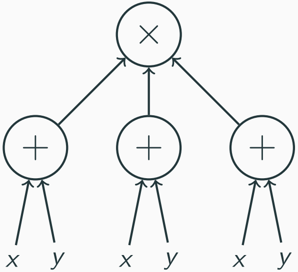

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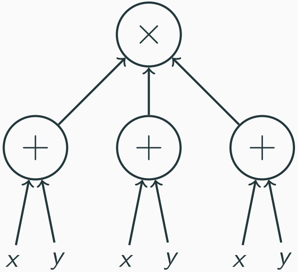

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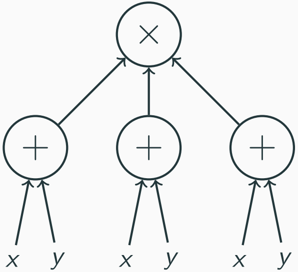

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Our Result: Any formula computing $\mathrm{ESYM}_{n,0.1n}(\mathbf{x})$ has atleast $\Omega(n^2)$ vertices, where

$$\mathrm{ESYM}_{n,0.1n}(\mathbf{x}) = \sum_{i_1 < \cdots < i_{0.1n} \in [n]} \prod_{j=1}^{0.1n} x_{i_j}.$$

Some Subtelties: Why Multilinear?

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- Individual Degree of every variable is 1.
- Multilinearisation of the SY polynomial gives an $\Omega(n^2/\log n)$ lower bound.
- Kalorkoti's method can not give a better bound against multilinear polynomials.

Conclusion

Proving Lower Bounds is Cool!

You should do it too... :)

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Thank You !

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