# Separating ABPs and some Structured Formulas in the Non-Commutative Setting

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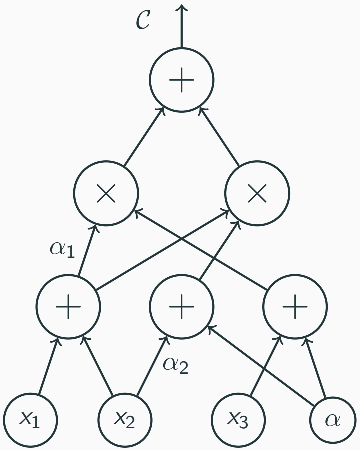
# Algebraic Models of Computation



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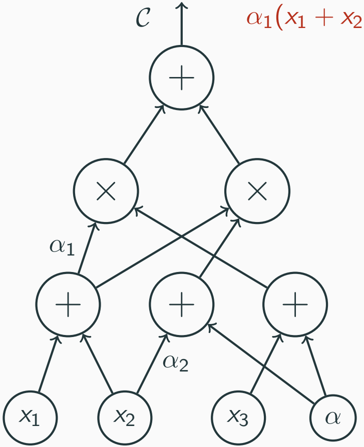


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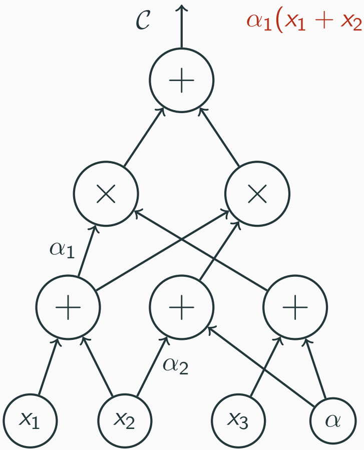


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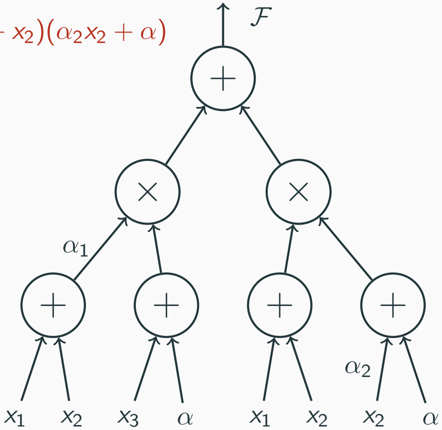

# Lower Bounds in Algebraic Circuit Complexity

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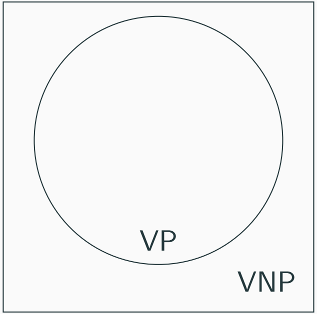


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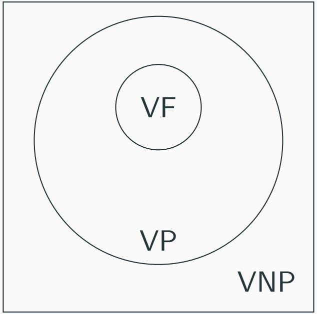


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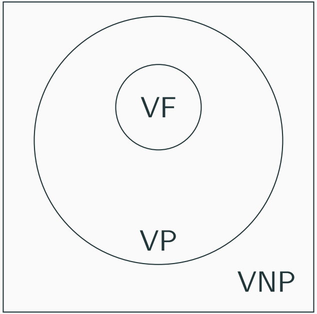


Are the inclusions tight?

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# The Current Best Bounds

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**Note**: There is a circuit of size $O(n \log^2 n)$ computing $\mathrm{ESYM}_{n,0.1n}(\mathbf{x})$. Therefore this shows a super-linear separation between formulas and circuits for a multilinear polynomial.

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$$f(x, y) = (x + y) \times (x + y) = x^2 + xy + yx + y^2 \neq x^2 + 2xy + y^2$$

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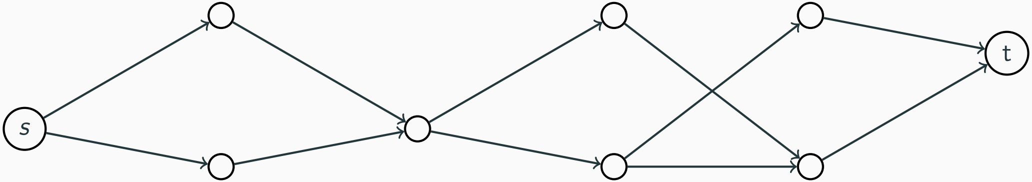
$$\mathrm{VF}_{\mathrm{nc}} \subseteq \mathrm{VBP}_{\mathrm{nc}} \subseteq \mathrm{VP}_{\mathrm{nc}}$$

So Nisan actually showed that $\mathrm{VBP}_{\mathrm{nc}} \neq \mathrm{VP}_{\mathrm{nc}}$.

# Algebraic Branching Programs



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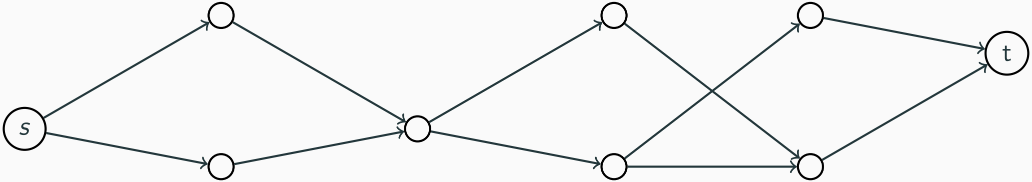


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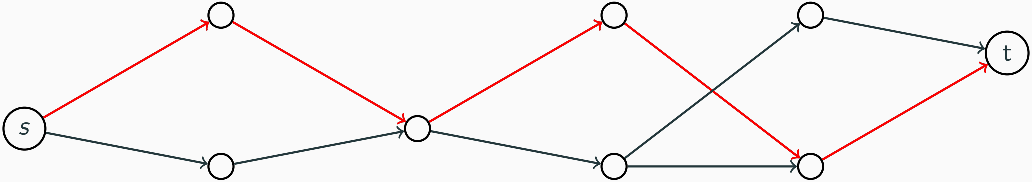


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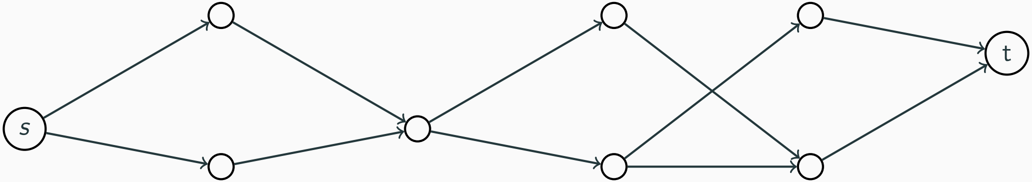


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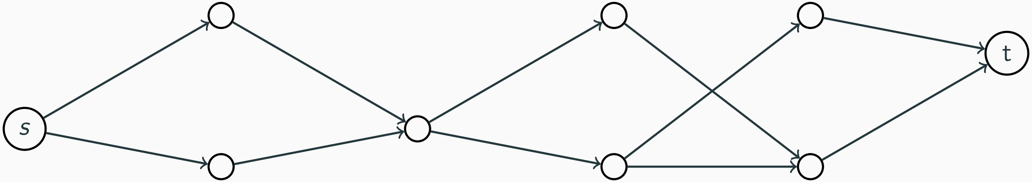


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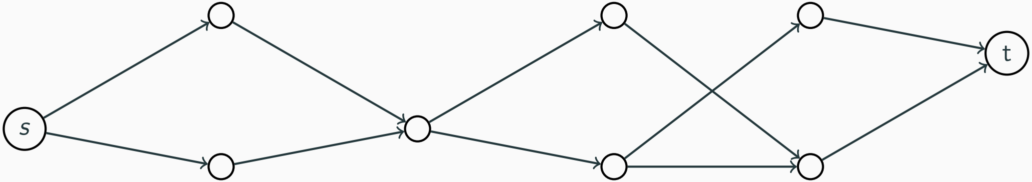


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# The ABP vs Formulas Question

**Nisan**: Any ABP computing $\mathrm{Pal}_n(x_0, x_1) = \sum_{w \in \{0,1\}^{n/2}} \mathbf{x}_w \cdot \mathbf{x}_{w^R}$ has size $2^{\Omega(n)}$.

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In comparison, the best lower bound against ABPs in the commutative setting is just quadratic.

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**The Question**[Nisan]: Is $\mathrm{VBP}_{\mathrm{nc}} = \mathrm{VF}_{\mathrm{nc}}$?

# Possible Approaches

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We only know how to multilinearise formulas when the degree is small [Raz].

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**[Tavenas, Limaye, Srinivasan]**

Any homogeneous non-commutative formula computing $\mathrm{IMM}_{n,n}$ must have size $n^{\Omega(\log \log n)}$.

# Main Result

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**Main Result**:

There is a tight superpolynomial separation between *abecedarian* formulas and ABPs.

# Abecedarian Polynomials

Generalises the notion of ordered polynomials ([Hrubes-Wigderson-Yehudayoff]).

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Generalises the notion of ordered polynomials ([Hrubes-Wigderson-Yehudayoff]).

$$\mathrm{Det}_n(\mathbf{x}) = \sum_{\sigma \in S_n} (-1)^{\mathrm{sgn}(\sigma)} x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$$

| **Buckets** | **Example** |
|---|---|
| | |
| $\{X_i\}_{i \in [n]}$ where $X_i = \{x_{ij}\}_{j \in [n]}$ | $\mathrm{Det}_n(\mathbf{x})$ |
| | |

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$$\mathrm{CHSYM}_{n,d}(\mathbf{x}) = \sum_{1 \leq i_1 \leq \ldots \leq i_d \leq n} x_{i_1} \cdots x_{i_d}$$

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Variables in every monomial arranged in non-decreasing order of bucket indices.

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$$\mathrm{ESYM}_{n,d}(\mathbf{x}) = \sum_{1 \le i_1 < \ldots < i_d \le n} x_{i_1} \cdots x_{i_d}$$

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$$f(\mathbf{x}) \xrightarrow[\text{in ascending order}]{\text{Order the monomials}} f^{(nc)}(\mathbf{x})$$

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**Note**:

$$\mathrm{ESYM}^{(\mathrm{ord})}_{n,d} = \sum_{1\le i_1<\ldots<i_d\le n} x^{(1)}_{i_1}\cdots x^{(d)}_{i_d}$$

is abecedarian w.r.t. both $\left\{X_k = \left\{x_i^{(k)}\right\}_{i\in[n]}\right\}_{k\in[d]}$ as well as $\left\{X_i = \left\{x_i^{(k)}\right\}_{k\in[d]}\right\}_{i\in[n]}$.

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Abecedarian Formulas: Non-commutative formulas with a syntactic restriction that makes them naturally compute abecedarian polynomials.

# The Explicit Statement

$$\text{linked\_CHSYM}_{n,d}(\mathbf{x}) = \sum_{i_0=1}^{n} \left( \sum_{i_0 \le i_1 \le \ldots \le i_d \le n} x_{i_0,i_1} \cdot x_{i_1,i_2} \cdots x_{i_{d-1},i_d} \right)$$

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If a formula of size $s$ computes a polynomial that is abecedarian with respect to a partition of size $O(\log n)$, then it can be converted into an abecedarian formula of size $\text{poly}(s)$.

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If there is a homogeneous structured abecedarian formula of size $s$ computing $h_{n/2,d}(\mathbf{x})$ and a homogeneous abecedarian formula of size $s'$ computing $\mathrm{CHSYM}_{n/2,d'}(\mathbf{x})$, then there is a homogeneous abecedarian formula computing $\mathrm{CHSYM}_{n/2,d\cdot d'}(\mathbf{x})$ of size $s \cdot s'$.

- There is a small homogeneous abecedarian formula computing $\mathrm{CHSYM}_{n/2,n/2}(\mathbf{x})$.
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**Note**: The last step uses ideas similar to those used by Raz to *multilinearise* formulas. This is why the transformation is efficient only when the number of buckets in the partition is small.

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- Can we prove a super-polynomial lower bound against abecedarian formulas for a polynomial when the partition size is $O(\log n)$?
- Can ideas from [Raz] or [DMPY] be modified to work for the non-commutative setting?

**Thank you!**