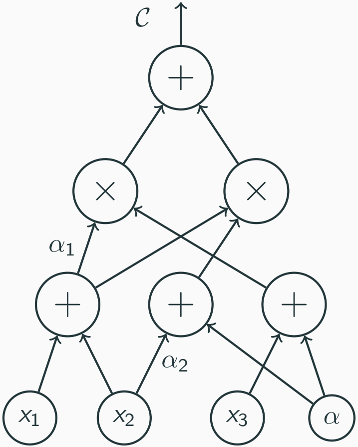
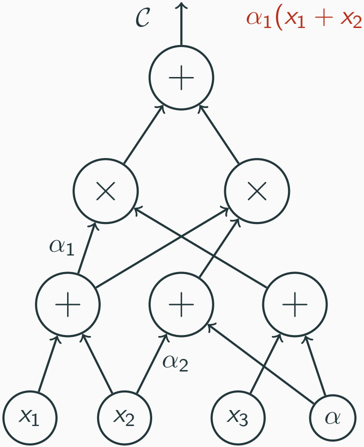
A Quadratic Lower Bound against Homogeneous Non-Commutative Circuits

Prerona Chatterjee (joint work with Pavel Hrubeš)
Institute of Mathematics, Czech Academy of Sciences

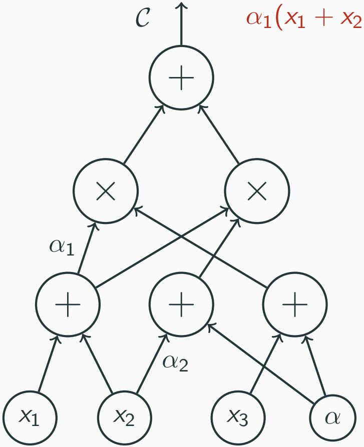
September 16, 2022

Algebraic Circuits





$\alpha_1(x_1 + x_2)(x_3 + \alpha) + (x_1 + x_2)(\alpha_2 x_2 + \alpha)$

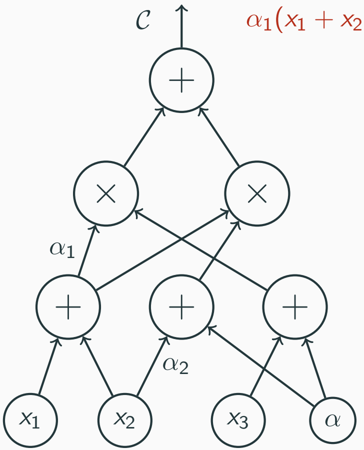


$\alpha_1(x_1 + x_2)(x_3 + \alpha) + (x_1 + x_2)(\alpha_2 x_2 + \alpha)$

Objects of Study

Polynomials over n variables of degree d.

Algebraic Circuits



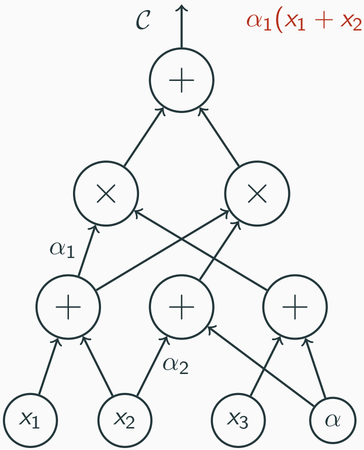
$\alpha_1(x_1 + x_2)(x_3 + \alpha) + (x_1 + x_2)(\alpha_2 x_2 + \alpha)$

Objects of Study

Polynomials over n variables of degree d.

Central Question: Find explicit polynomials that cannot be computed by efficient circuits.

Algebraic Circuits



$\alpha_1(x_1 + x_2)(x_3 + \alpha) + (x_1 + x_2)(\alpha_2 x_2 + \alpha)$

Objects of Study

Polynomials over n variables of degree d.

Central Question: Find explicit polynomials that cannot be computed by efficient circuits.

[Baur-Strassen]: Any algebraic circuit computing $\sum_{i=1}^{n} x_i^d$ requires $\Omega(n \log d)$ wires.

The Non-Commutative Setting

$$f(x, y) = (x + y) \times (x + y) = x^2 + xy + yx + y^2 \neq x^2 + 2xy + y^2$$

The Non-Commutative Setting

$$f(x, y) = (x + y) \times (x + y) = x^2 + xy + yx + y^2 \neq x^2 + 2xy + y^2$$

Non-Commutative Circuits: The multiplication gates, additionally, respect the order.

The Non-Commutative Setting

$$f(x, y) = (x + y) \times (x + y) = x^2 + xy + yx + y^2 \neq x^2 + 2xy + y^2$$

Non-Commutative Circuits: The multiplication gates, additionally, respect the order.

Can we do something better in this setting?

We should be able to...

[Nisan]: Exponential lower bound against NC ABPs and formulas.

The best known lower bound against general ABPs, formulas is quadratic [C-Kumar-She-Volk].

We should be able to...

[Nisan]: Exponential lower bound against NC ABPs and formulas.

The best known lower bound against general ABPs, formulas is quadratic [C-Kumar-She-Volk].

[Tavenas-Limaye-Srinivasan]: Super-polynomial separation between homogeneous NC formulas and ABPs.

We should be able to...

[Nisan]: Exponential lower bound against NC ABPs and formulas.

The best known lower bound against general ABPs, formulas is quadratic [C-Kumar-She-Volk].

[Tavenas-Limaye-Srinivasan]: Super-polynomial separation between homogeneous NC formulas and ABPs.

[Carmossino-Impagliazzo-Lovett-Mihajlin]: Super-linear lower bounds against non-commutative circuits for constant degree polynomials imply exponential lower bounds.

We should be able to...

[Nisan]: Exponential lower bound against NC ABPs and formulas.

The best known lower bound against general ABPs, formulas is quadratic [C-Kumar-She-Volk].

[Tavenas-Limaye-Srinivasan]: Super-polynomial separation between homogeneous NC formulas and ABPs.

[Carmossino-Impagliazzo-Lovett-Mihajlin]: Super-linear lower bounds against non-commutative circuits for constant degree polynomials imply exponential lower bounds.

No such result known in the general setting.

Our Result

The best lower bound against NC circuits continues to be $\Omega(n \log d)$.

Our Result

The best lower bound against NC circuits continues to be $\Omega(n \log d)$.

Can we at least say something in the homogeneous case?

Our Result

The best lower bound against NC circuits continues to be $\Omega(n \log d)$.

Can we at least say something in the homogeneous case?

Theorem: Any homogeneous non-commutative circuit computing

$$\mathrm{ESYM}_{n,d} = \sum_{1 \le i_1 < \cdots < i_d \le n} x_{i_1} \cdots x_{i_d}$$

has size $\Omega(nd')$ where $d' = \min(d, n-d)$.

A simple proof of an obvious fact

A simple proof of an obvious fact

f: Homogeneous non-commutative polynomial of degree d.

A simple proof of an obvious fact

f: Homogeneous non-commutative polynomial of degree d.

$f^{(i)}$: Polynomial got from f by setting variables in positions other than i, $i+1$ to 1.

A simple proof of an obvious fact

f: Homogeneous non-commutative polynomial of degree d.
$f^{(i)}$: Polynomial got from f by setting variables in positions other than i, $i+1$ to 1.

$$\mu(f) = \text{rank}\left(\text{span}_{\mathbb{F}}\left(\{1\} \cup \left\{f^{(0)}, f^{(1)}, \ldots, f^{(d)}\right\} \setminus \{0\}\right)\right).$$

A simple proof of an obvious fact

f: Homogeneous non-commutative polynomial of degree d.
$f^{(i)}$: Polynomial got from f by setting variables in positions other than i, $i+1$ to 1.

$$\mu(f) = \operatorname{rank}\left(\operatorname{span}_{\mathbb{F}}\left(\{1\} \cup \left\{f^{(0)}, f^{(1)}, \ldots, f^{(d)}\right\} \setminus \{0\}\right)\right).$$

$\mathcal{C}$: Homogeneous non-commutative circuit.

$$\mu(\mathcal{C}) = \operatorname{rank}\left(\operatorname{span}_{\mathbb{F}}\left(\bigcup_{g \in \mathcal{C}} \left\{g^{(0)}, g^{(1)}, \ldots, g^{(d)}\right\}\right)\right).$$

A simple proof of an obvious fact

$g^{(i)}$: Polynomial got from g by setting variables in positions other than i, $i+1$ to 1.

$$\mu(g) = \text{rank}\left(\text{span}_{\mathbb{F}}\left(\{1\} \cup \left\{g^{(0)}, g^{(1)}, \ldots, g^{(d)}\right\} \setminus \{0\}\right)\right).$$

A simple proof of an obvious fact

$g^{(i)}$: Polynomial got from g by setting variables in positions other than i, $i+1$ to 1.

$$\mu(g) = \text{rank}\left(\text{span}_{\mathbb{F}}\left(\{1\} \cup \left\{g^{(0)}, g^{(1)}, \ldots, g^{(d)}\right\} \setminus \{0\}\right)\right).$$

Claim 1: If $\mathcal{C}$ is a homogeneous non-commutative circuit of size s, then $\mu(\mathcal{C}) \leq s+1$.

A simple proof of an obvious fact

$g^{(i)}$: Polynomial got from g by setting variables in positions other than i, $i+1$ to 1.

$$\mu(g) = \text{rank}\left(\text{span}_{\mathbb{F}}\left(\{1\} \cup \left\{g^{(0)}, g^{(1)}, \ldots, g^{(d)}\right\} \setminus \{0\}\right)\right).$$

Claim 1: If $\mathcal{C}$ is a homogeneous non-commutative circuit of size s, then $\mu(\mathcal{C}) \leq s+1$.

Claim 2: For $f = x_1 \cdots x_n$, $\mu(f) = n+1$.

A simple proof of an obvious fact

$g^{(i)}$: Polynomial got from g by setting variables in positions other than i, $i+1$ to 1.

$$\mu(g) = \text{rank}\left(\text{span}_{\mathbb{F}}\left(\{1\} \cup \left\{g^{(0)}, g^{(1)}, \ldots, g^{(d)}\right\} \setminus \{0\}\right)\right).$$

Claim 1: If $\mathcal{C}$ is a homogeneous non-commutative circuit of size s, then $\mu(\mathcal{C}) \leq s+1$.

Claim 2: For $f = x_1 \cdots x_n$, $\mu(f) = n+1$.

Therefore $s \geq n$.

Using it to prove a "not so obvious" fact

The same technique shows: Any homogeneous non-commutative circuit computing

$$(x_0 \cdots x_0 x_0)(x_0 \cdots x_0 x_1)(x_0 \cdots x_1 x_0) \cdots \cdots (x_1 \cdots x_1 x_1)$$

must have size $\Omega\left(\frac{d}{\log d}\right)$.

Using it to prove a “not so obvious” fact

The same technique shows: Any homogeneous non-commutative circuit computing

$$(x_0 \cdots x_0 x_0)(x_0 \cdots x_0 x_1)(x_0 \cdots x_1 x_0) \cdots\cdots (x_1 \cdots x_1 x_1)$$

must have size $\Omega\left(\frac{d}{\log d}\right)$.

How can non-homogeneity possibly help in computing a monomial?

Using it to prove a “not so obvious” fact

The same technique shows: Any homogeneous non-commutative circuit computing

$$(x_0 \cdots x_0 x_0)(x_0 \cdots x_0 x_1)(x_0 \cdots x_1 x_0) \cdots\cdots (x_1 \cdots x_1 x_1)$$

must have size $\Omega\left(\frac{d}{\log d}\right)$.

How can non-homogeneity possibly help in computing a monomial?

Question: Can we prove the same lower bound against general non-commutative circuits?

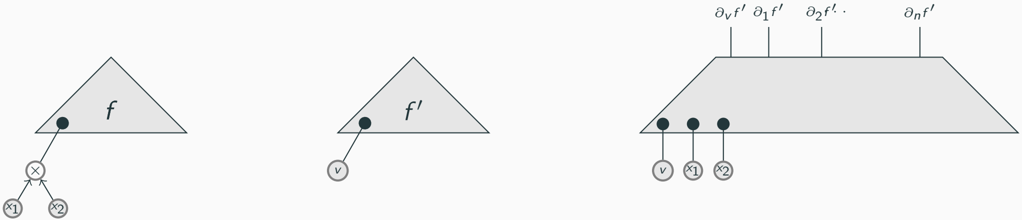
Getting back to the quadratic lower bound

[Baur-Strassen]: If there is a circuit of size s computing $f \in \mathbb{F}[\mathbf{x}]$, then there is a circuit of size at most $5s$ that simultaneously compute $\{\partial_{x_1} f, \partial_{x_2} f, \ldots, \partial_{x_n} f\}$.

Getting back to the quadratic lower bound

[Baur-Strassen]: If there is a circuit of size s computing $f \in \mathbb{F}[\mathbf{x}]$, then there is a circuit of size at most $5s$ that simultaneously compute $\{\partial_{x_1} f, \partial_{x_2} f, \ldots, \partial_{x_n} f\}$.

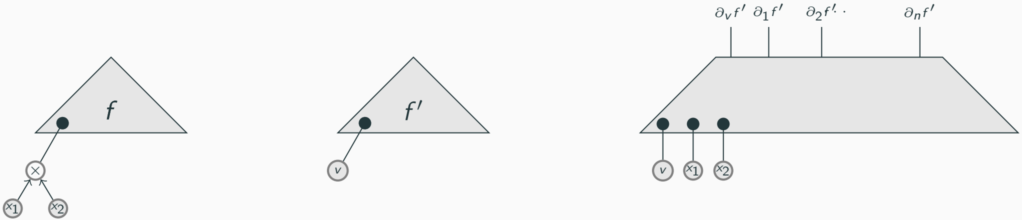
Step 1:



Getting back to the quadratic lower bound

[Baur-Strassen]: If there is a circuit of size s computing $f \in \mathbb{F}[\mathbf{x}]$, then there is a circuit of size at most $5s$ that simultaneously compute $\{\partial_{x_1} f, \partial_{x_2} f, \ldots, \partial_{x_n} f\}$.

Step 1:

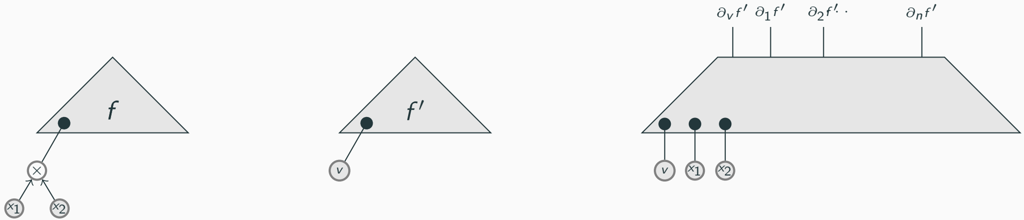


Step 2: Use chain rule.

Getting back to the quadratic lower bound

[Baur-Strassen]: If there is a circuit of size s computing $f \in \mathbb{F}[\mathbf{x}]$, then there is a circuit of size at most $5s$ that simultaneously compute $\{\partial_{x_1} f, \partial_{x_2} f, \ldots, \partial_{x_n} f\}$.

Step 1:



Step 2: Use chain rule. Exists in the NC setting as well if we work with position 1.

Making [Baur-Strassen] work in the homogeneous setting

Target: If there is a homogeneous circuit of size s computing $f \in \mathbb{F}[\mathbf{x}]$, then there is a homogeneous circuit of size at most $5s$ that simultaneously compute $\{\partial_{x_1} f, \partial_{x_2} f, \ldots, \partial_{x_n} f\}$.

Making [Baur-Strassen] work in the homogeneous setting

Target: If there is a homogeneous circuit of size s computing $f \in \mathbb{F}[\mathbf{x}]$, then there is a homogeneous circuit of size at most $5s$ that simultaneously compute $\{\partial_{x_1} f, \partial_{x_2} f, \ldots, \partial_{x_n} f\}$.

Weights: $w_i = \mathrm{wt}(x_i)$.

Making [Baur-Strassen] work in the homogeneous setting

Target: If there is a homogeneous circuit of size s computing $f \in \mathbb{F}[\mathbf{x}]$, then there is a homogeneous circuit of size at most $5s$ that simultaneously compute $\{\partial_{x_1} f, \partial_{x_2} f, \ldots, \partial_{x_n} f\}$.

Weights: $w_i = \mathrm{wt}(x_i)$. Given $\mathbf{w} = (w_1, \ldots, w_n)$, define $\mathbf{w}$-homogeneous.

Making [Baur-Strassen] work in the homogeneous setting

Target: If there is a homogeneous circuit of size s computing $f \in \mathbb{F}[\mathbf{x}]$, then there is a homogeneous circuit of size at most $5s$ that simultaneously compute $\{\partial_{x_1} f, \partial_{x_2} f, \ldots, \partial_{x_n} f\}$.

Weights: $w_i = \mathrm{wt}(x_i)$. Given $\mathbf{w} = (w_1, \ldots, w_n)$, define $\mathbf{w}$-homogeneous.

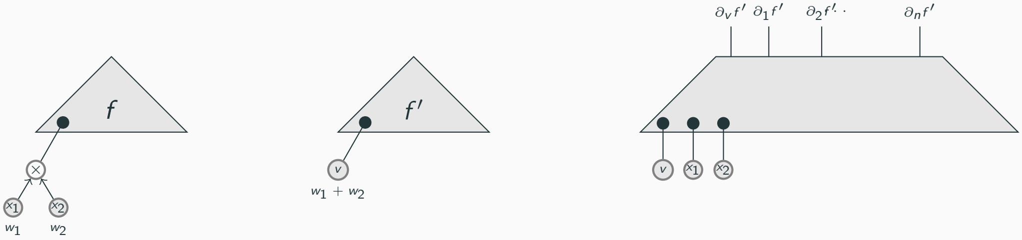
Lemma: If there is a $\mathbf{w}$-homogeneous circuit of size s computing $f \in \mathbb{F}[\mathbf{x}]$, then there is a $\mathbf{w}$-homogeneous circuit of size at most $5s$ that simultaneously compute $\{\partial_{x_1} f, \partial_{x_2} f, \ldots, \partial_{x_n} f\}$.

Making [Baur-Strassen] work in the homogeneous setting

Target: If there is a homogeneous circuit of size s computing $f \in \mathbb{F}[\mathbf{x}]$, then there is a homogeneous circuit of size at most $5s$ that simultaneously compute $\{\partial_{x_1} f, \partial_{x_2} f, \ldots, \partial_{x_n} f\}$.

Weights: $w_i = \text{wt}(x_i)$. Given $\mathbf{w} = (w_1, \ldots, w_n)$, define $\mathbf{w}$-homogeneous.

Lemma: If there is a $\mathbf{w}$-homogeneous circuit of size s computing $f \in \mathbb{F}[\mathbf{x}]$, then there is a $\mathbf{w}$-homogeneous circuit of size at most $5s$ that simultaneously compute $\{\partial_{x_1} f, \partial_{x_2} f, \ldots, \partial_{x_n} f\}$.



Recalling the measure

f: Homogeneous non-commutative polynomial of degree d.

Recalling the measure

f: Homogeneous non-commutative polynomial of degree d.

$f^{(i)}$: Polynomial got from f by setting variables in positions other than i, $i+1$ to 1.

Recalling the measure

f: Homogeneous non-commutative polynomial of degree d.

$f^{(i)}$: Polynomial got from f by setting variables in positions other than i, $i+1$ to 1.

$$\mu(f) = \text{rank}\left(\text{span}_{\mathbb{F}}\left(\{1\} \cup \left\{f^{(0)}, f^{(1)}, \ldots, f^{(d)}\right\} \setminus \{0\}\right)\right).$$

Recalling the measure

f: Homogeneous non-commutative polynomial of degree d.
$f^{(i)}$: Polynomial got from f by setting variables in positions other than i, $i+1$ to 1.

$$\mu(f) = \text{rank}\left(\text{span}_{\mathbb{F}}\left(\{1\} \cup \left\{f^{(0)}, f^{(1)}, \ldots, f^{(d)}\right\} \setminus \{0\}\right)\right).$$

$\mathcal{C}$: Homogeneous non-commutative circuit.

$$\mu(\mathcal{C}) = \text{rank}\left(\text{span}_{\mathbb{F}}\left(\bigcup_{g \in \mathcal{C}} \left\{g^{(0)}, g^{(1)}, \ldots, g^{(d)}\right\}\right)\right).$$

Recalling the measure

f: Homogeneous non-commutative polynomial of degree d.
$f^{(i)}$: Polynomial got from f by setting variables in positions other than i, $i+1$ to 1.

$$\mu(f) = \text{rank}\left(\text{span}_{\mathbb{F}}\left(\{1\} \cup \left\{f^{(0)}, f^{(1)}, \ldots, f^{(d)}\right\} \setminus \{0\}\right)\right).$$

$\mathcal{C}$: Homogeneous non-commutative circuit.

$$\mu(\mathcal{C}) = \text{rank}\left(\text{span}_{\mathbb{F}}\left(\bigcup_{g \in \mathcal{C}} \left\{g^{(0)}, g^{(1)}, \ldots, g^{(d)}\right\}\right)\right).$$

$$\mu(\text{out}(\mathcal{C})) = \text{rank}\left(\text{span}_{\mathbb{F}}\left(\bigcup_{g \in \text{out}(\mathcal{C})} \left\{g^{(0)}, g^{(1)}, \ldots, g^{(d)}\right\}\right)\right).$$

Completing the proof

$\mathcal{C}$: Homogeneous circuit of size s computing f.

Completing the proof

$\mathcal{C}$: Homogeneous circuit of size s computing f.

$\mathcal{C}'$: Homogeneous circuit of size $5s$ that simultaneously compute $\{\partial_{1,x_1} f, \partial_{1,x_2} f, \ldots, \partial_{1,x_n} f\}$.

Completing the proof

$\mathcal{C}$: Homogeneous circuit of size s computing f.

$\mathcal{C}'$: Homogeneous circuit of size $5s$ that simultaneously compute $\{\partial_{1,x_1} f, \partial_{1,x_2} f, \ldots, \partial_{1,x_n} f\}$.

$$\mu(\mathcal{C}') \leq 5s + 1$$

Completing the proof

$\mathcal{C}$: Homogeneous circuit of size s computing f.

$\mathcal{C}'$: Homogeneous circuit of size $5s$ that simultaneously compute $\{\partial_{1,x_1} f, \partial_{1,x_2} f, \ldots, \partial_{1,x_n} f\}$.

$$\mu(\mathcal{C}') \leq 5s + 1$$

Task: Find n-variate, degree d, f such that if $\text{out}(\mathcal{C}) = \mathbb{D}(f) = \{\partial_{1,x_1} f, \partial_{1,x_2} f, \ldots, \partial_{1,x_n} f\}$, then

$$\mu(\text{out}(\mathcal{C}')) = \Omega(nd).$$

Completing the proof

$\mathcal{C}$: Homogeneous circuit of size s computing f.

$\mathcal{C}'$: Homogeneous circuit of size $5s$ that simultaneously compute $\{\partial_{1,x_1} f, \partial_{1,x_2} f, \ldots, \partial_{1,x_n} f\}$.

$$\mu(\mathcal{C}') \leq 5s + 1$$

Task: Find n-variate, degree d, f such that if $\text{out}(\mathcal{C}) = \mathbb{D}(f) = \{\partial_{1,x_1} f, \partial_{1,x_2} f, \ldots, \partial_{1,x_n} f\}$, then

$$\mu(\text{out}(\mathcal{C}')) = \Omega(nd).$$

Use the fact that $\mu(\text{out}(\mathcal{C}')) \leq \mu(\mathcal{C})$ to complete the proof.

Polynomial with a large measure

$$f = \mathrm{ESYM}_{n,\frac{n}{2}+1}(\mathbf{x}) = \sum_{1 \le i_1 < \cdots < i_{\frac{n}{2}+1} \le n} \left(\prod_{j=1}^{\frac{n}{2}+1} x_{i_j} \right)$$

Polynomial with a large measure

$x_{\frac{n}{2}+1}x_{\frac{n}{2}+2} \quad \cdots \quad x_2 x_{\frac{n}{2}+2} \quad \cdots \quad \cdots \quad x_{n-2}x_{n-1} \quad \cdots \quad x_{\frac{n}{2}-1}x_{n-1} \quad x_{n-1}x_n \quad \cdots \quad x_{\frac{n}{2}}x_n$

$(1, \partial_{1,x_{\frac{n}{2}}} f)$

$\vdots$

$(1, \partial_{1,x_1} f)$

$\vdots$

$\vdots$

$(\frac{n}{2} - 2, \partial_{1,x_{\frac{n}{2}}} f)$

$\vdots$

$(\frac{n}{2} - 2, \partial_{1,x_1} f)$

$(\frac{n}{2} - 1, \partial_{1,x_{\frac{n}{2}}} f)$

$\vdots$

$(\frac{n}{2} - 1, \partial_{1,x_1} f)$

$$f = \mathrm{ESYM}_{n,\frac{n}{2}+1}(\mathbf{x}) = \sum_{1 \le i_1 < \cdots < i_{\frac{n}{2}+1} \le n} \left(\prod_{j=1}^{\frac{n}{2}+1} x_{i_j} \right)$$

Polynomial with a large measure

$x_{\frac{n}{2}+1}x_{\frac{n}{2}+2} \quad \cdots \quad x_2 x_{\frac{n}{2}+2} \quad \cdots \quad \cdots \quad x_{n-2}x_{n-1} \quad \cdots \quad x_{\frac{n}{2}-1}x_{n-1} \quad x_{n-1}x_n \quad \cdots \quad x_{\frac{n}{2}}x_n$

$(1, \partial_{1,x_{\frac{n}{2}}} f)$

$\vdots$

$(1, \partial_{1,x_1} f)$

$\vdots$

$\vdots$

$(\frac{n}{2} - 2, \partial_{1,x_{\frac{n}{2}}} f)$

$\vdots$

$(\frac{n}{2} - 2, \partial_{1,x_1} f)$

$(\frac{n}{2} - 1, \partial_{1,x_{\frac{n}{2}}} f)$

$\vdots$

$(\frac{n}{2} - 1, \partial_{1,x_1} f)$

$$f = \mathrm{ESYM}_{n,\frac{n}{2}+1}(\mathbf{x}) = \sum_{1 \le i_1 < \cdots < i_{\frac{n}{2}+1} \le n} \left(\prod_{j=1}^{\frac{n}{2}+1} x_{i_j} \right)$$

Defining the matrix $\mathcal{M}(f)$

$x_i x_j$

$(k, \partial_{1,x_\ell} f)$ $\quad$ $\boxed{(\partial_{1,x_\ell} f)^{(k)}}$

Polynomial with a large measure

$x_{\frac{n}{2}+1}x_{\frac{n}{2}+2}$ $\cdots$ $x_2x_{\frac{n}{2}+2}$ $\cdots$ $\cdots$ $x_{n-2}x_{n-1}$ $\cdots$ $x_{\frac{n}{2}-1}x_{n-1}$ $x_{n-1}x_n$ $\cdots$ $x_{\frac{n}{2}}x_n$

$(1, \partial_{1,x_{\frac{n}{2}}} f)$

$\vdots$

$(1, \partial_{1,x_1} f)$

$\vdots$

$\vdots$

$(\frac{n}{2} - 2, \partial_{1,x_{\frac{n}{2}}} f)$

$\vdots$

$(\frac{n}{2} - 2, \partial_{1,x_1} f)$

$(\frac{n}{2} - 1, \partial_{1,x_{\frac{n}{2}}} f)$

$\vdots$

$(\frac{n}{2} - 1, \partial_{1,x_1} f)$

$$f = \mathrm{ESYM}_{n,\frac{n}{2}+1}(\mathbf{x}) = \sum_{1 \le i_1 < \cdots < i_{\frac{n}{2}+1} \le n} \left(\prod_{j=1}^{\frac{n}{2}+1} x_{i_j} \right)$$

Defining the matrix $\mathcal{M}(f)$

$x_i x_j$

$(k, \partial_{1,x_\ell} f)$ $\quad$ $(\partial_{1,x_\ell} f)^{(k)}$

$$\mu(\mathbb{D}(f)) \ge \mathrm{rank}(\mathcal{M}(f))$$

Polynomial with a large measure

$x_{\frac{n}{2}+1}x_{\frac{n}{2}+2} \quad \cdots \quad x_2 x_{\frac{n}{2}+2} \quad \cdots \quad \cdots \quad x_{n-2}x_{n-1} \quad \cdots \quad x_{\frac{n}{2}-1}x_{n-1} \quad x_{n-1}x_n \quad \cdots \quad x_{\frac{n}{2}}x_n$

$(1, \partial_{1,x_{\frac{n}{2}}} f)$

$\vdots$

$(1, \partial_{1,x_1} f)$

$\vdots$

$\vdots$

$(\frac{n}{2} - 2, \partial_{1,x_{\frac{n}{2}}} f)$

$\vdots$

$(\frac{n}{2} - 2, \partial_{1,x_1} f)$

$(\frac{n}{2} - 1, \partial_{1,x_{\frac{n}{2}}} f)$

$\vdots$

$(\frac{n}{2} - 1, \partial_{1,x_1} f)$

$$f = \mathrm{ESYM}_{n,\frac{n}{2}+1}(\mathbf{x}) = \sum_{1 \le i_1 < \cdots < i_{\frac{n}{2}+1} \le n} \left(\prod_{j=1}^{\frac{n}{2}+1} x_{i_j} \right)$$

Defining the matrix $\mathcal{M}(f)$

$x_i x_j$

$(k, \partial_{1,x_\ell} f)$ $\quad$ $(\partial_{1,x_\ell} f)^{(k)}$

This matrix is lower triangular.

$$\mu(\mathbb{D}(f)) \ge \mathrm{rank}(\mathcal{M}(f))$$

Polynomial with a large measure

$x_{\frac{n}{2}+1}x_{\frac{n}{2}+2} \quad \cdots \quad x_2 x_{\frac{n}{2}+2} \quad \cdots \quad \cdots \quad x_{n-2}x_{n-1} \quad \cdots \quad x_{\frac{n}{2}-1}x_{n-1} \quad x_{n-1}x_n \quad \cdots \quad x_{\frac{n}{2}}x_n$

$(1, \partial_{1,x_{\frac{n}{2}}} f)$

$\vdots$

$(1, \partial_{1,x_1} f)$

$\vdots$

$\vdots$

$(\frac{n}{2} - 2, \partial_{1,x_{\frac{n}{2}}} f)$

$\vdots$

$(\frac{n}{2} - 2, \partial_{1,x_1} f)$

$(\frac{n}{2} - 1, \partial_{1,x_{\frac{n}{2}}} f)$

$\vdots$

$(\frac{n}{2} - 1, \partial_{1,x_1} f)$

$$f = \mathrm{ESYM}_{n,\frac{n}{2}+1}(\mathbf{x}) = \sum_{1 \le i_1 < \cdots < i_{\frac{n}{2}+1} \le n} \left(\prod_{j=1}^{\frac{n}{2}+1} x_{i_j} \right)$$

Defining the matrix $\mathcal{M}(f)$

$x_i x_j$

$(k, \partial_{1,x_\ell} f)$ $\qquad$ $\boxed{(\partial_{1,x_\ell} f)^{(k)}}$

This matrix is lower triangular.

$$\mu(\mathbb{D}(f)) \ge \mathrm{rank}(\mathcal{M}(f)) = \Omega(n^2).$$

Upper bounds for computing the polynomial

There is a homogeneous non-commutative circuit of size $O(n^2)$ that computes $\mathrm{ESYM}_{n,\frac{n}{2}+1}(\mathbf{x})$.

Upper bounds for computing the polynomial

There is a homogeneous non-commutative circuit of size $O(n^2)$ that computes $\mathrm{ESYM}_{n,\frac{n}{2}+1}(\mathbf{x})$.

Can we do any better in the non-homogeneous case?

Upper bounds for computing the polynomial

There is a homogeneous non-commutative circuit of size $O(n^2)$ that computes $\mathrm{ESYM}_{n,\frac{n}{2}+1}(\mathbf{x})$.

Can we do any better in the non-homogeneous case? Yes!!

Upper bounds for computing the polynomial

There is a homogeneous non-commutative circuit of size $O(n^2)$ that computes $\mathrm{ESYM}_{n,\frac{n}{2}+1}(\mathbf{x})$.

Can we do any better in the non-homogeneous case? Yes!!

There is a non-commutative circuit of size $O(n \log^2 n)$ that computes all the elementary symmetric polynomials simultaneously.

Upper bounds for computing the polynomial

There is a homogeneous non-commutative circuit of size $O(n^2)$ that computes $\mathrm{ESYM}_{n,\frac{n}{2}+1}(\mathbf{x})$.

Can we do any better in the non-homogeneous case? Yes!!

There is a non-commutative circuit of size $O(n \log^2 n)$ that computes all the elementary symmetric polynomials simultaneously.

How?

Upper bounds for computing the polynomial

There is a homogeneous non-commutative circuit of size $O(n^2)$ that computes $\mathrm{ESYM}_{n,\frac{n}{2}+1}(\mathbf{x})$.

Can we do any better in the non-homogeneous case? Yes!!

There is a non-commutative circuit of size $O(n \log^2 n)$ that computes all the elementary symmetric polynomials simultaneously.

How? $\mathrm{ESYM}_{n,d} = \mathsf{coeff}_{t^d} \left(\prod_{i=1}^{n}(1 + tx_i)\right)$

Upper bounds for computing the polynomial

There is a homogeneous non-commutative circuit of size $O(n^2)$ that computes $\mathrm{ESYM}_{n,\frac{n}{2}+1}(\mathbf{x})$.

Can we do any better in the non-homogeneous case? Yes!!

There is a non-commutative circuit of size $O(n \log^2 n)$ that computes all the elementary symmetric polynomials simultaneously.

How? $\mathrm{ESYM}_{n,d} = \mathsf{coeff}_{t^d}\left(\prod_{i=1}^{n}(1 + tx_i)\right)$ Use FFT recursively $\log n$ times.

Open Questions

- $\Omega(d)$ lower bound against non-commutative circuits.

Open Questions

- $\Omega(d)$ lower bound against non-commutative circuits.
- [CILM]-type amplification statements, but in terms of degree.

Open Questions

- $\Omega(d)$ lower bound against non-commutative circuits.
- [CILM]-type amplification statements, but in terms of degree.

Thank you!