Monotone Classes Beyond VNP

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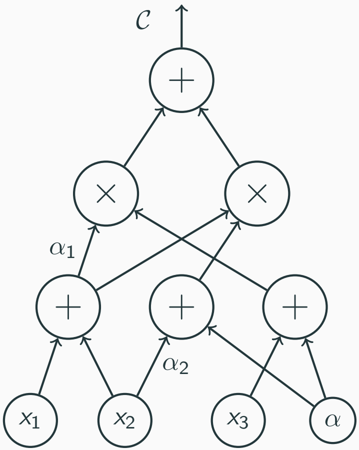
Tel Aviv University

August 16, 2023

Algebraic Classes Beyond VNP



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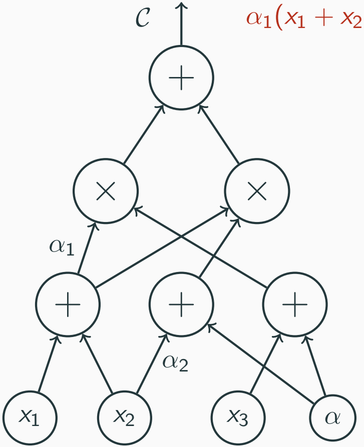

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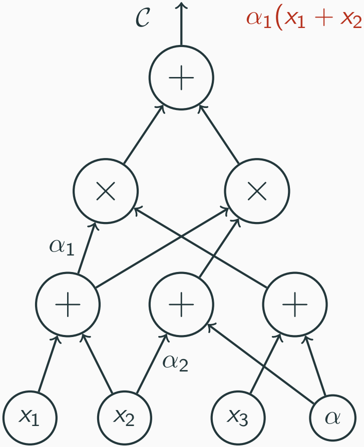

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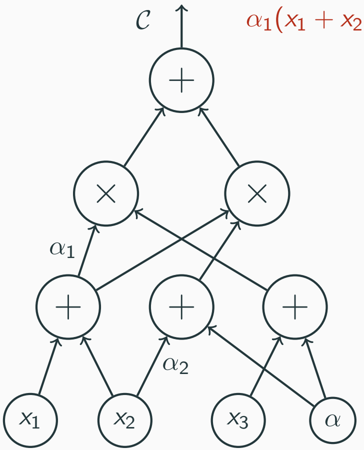

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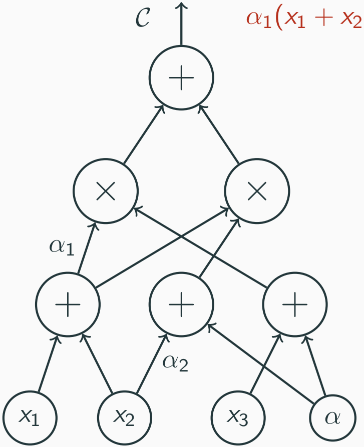

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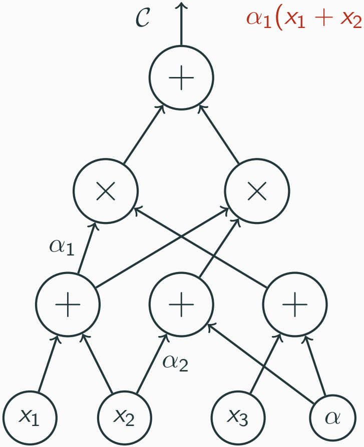

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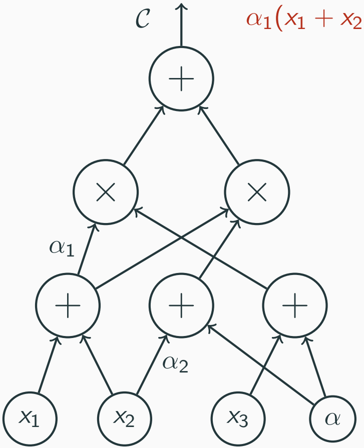

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What about classes beyond VNP?

The Class VPSPACE

[Koiran-Perifel]: $\{f_n\}_n$ is in $\mathsf{VPSPACE}^0$ if the following language is in PSPACE/ poly.

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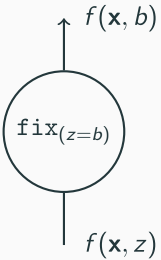

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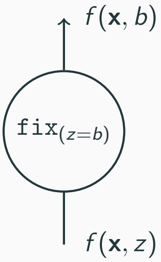

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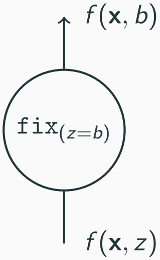

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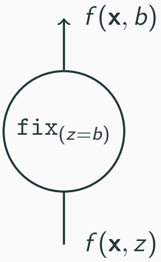

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Proof idea coming up in the next talk!

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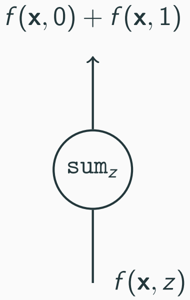

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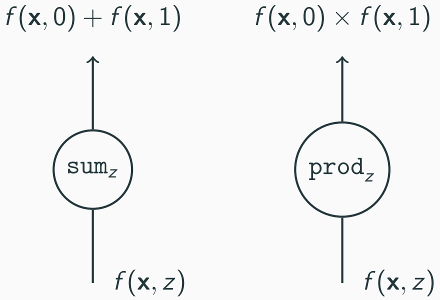

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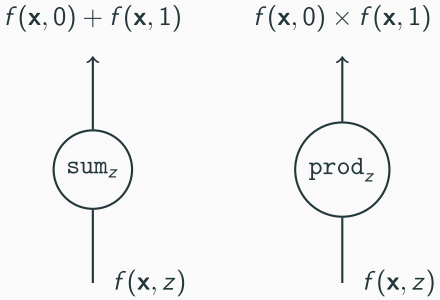

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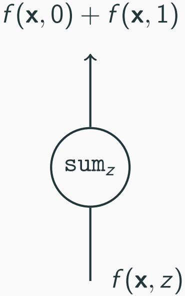

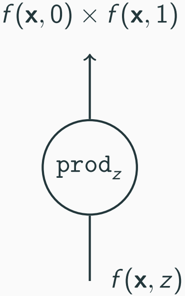

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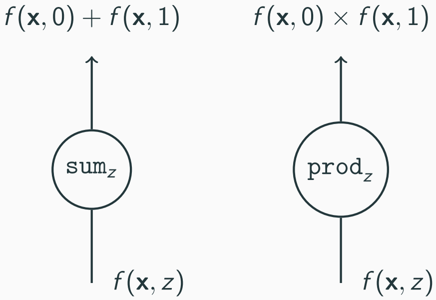

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Can we prove lower bounds in the monotone setting?

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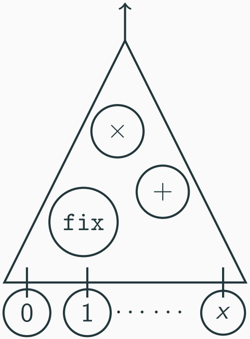

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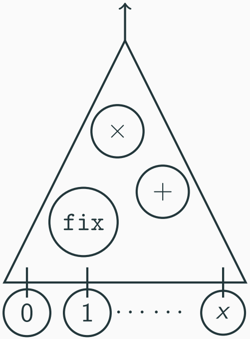

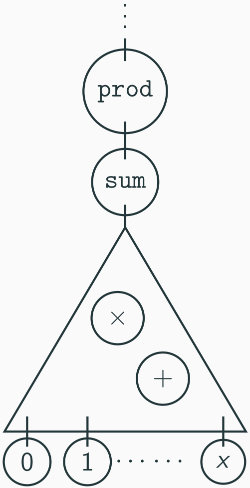

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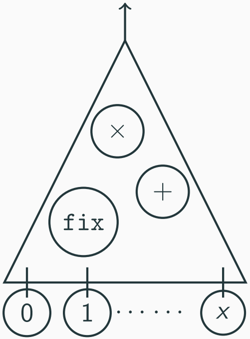
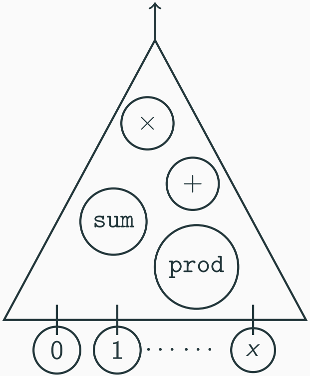
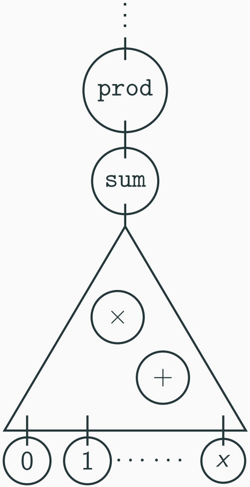
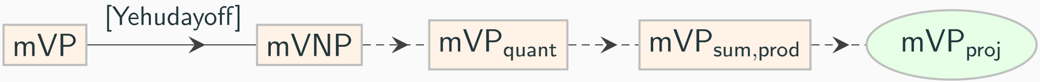






Our Results

Polynomials have bounded degree throughout



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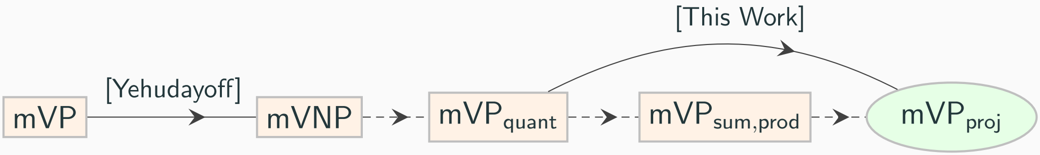

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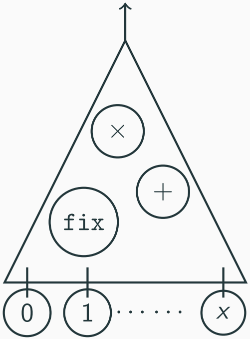

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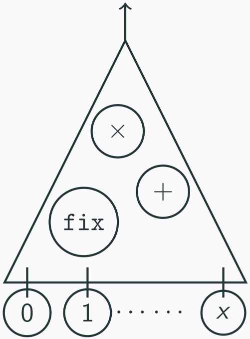

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$$P_0(\mathbf{x}, \mathbf{y}) := \left(\sum_{j=1}^{n} y_{1,j} x_{1,j} \right) \left(\sum_{j=1}^{n} y_{2,j} x_{2,j} \right) \cdots \left(\sum_{j=1}^{n} y_{n,j} x_{n,j} \right).$$

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Size: $\text{size}(P_0) = O(n^2)$ and $\text{size}(P_j) = \text{size}(P_{j-1}) + O(n^2) \implies \text{size}(P_n) = O(n^3)$.

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$$f \in \text{mVNP} \implies f = \sum_{\mathbf{z} \in \{0,1\}^m} g(\mathbf{x}, \mathbf{z}) \qquad \text{for } g(\mathbf{x}, \mathbf{z}) \in \text{mVP}$$

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[Yehudayoff]: "Support-based" lower bounds against mVP can be extended to mVNP.

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"Support-based" lower bounds against mVP can be extended to $\mathsf{mVP}_{\mathsf{quant}}$ if f is irreducible.

Open Questions I

- If $f \in \mathrm{mVP}_{\mathrm{quant}}$, then

$$f(\mathbf{x}) = \sum_{\mathbf{b} \in \{0,1\}^{|\mathbf{w}|}} A_f(\mathbf{w} = \mathbf{b}) \cdot g_f(\mathbf{x}, \mathbf{w} = \mathbf{b})$$

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- $\mathrm{VPH} = \mathtt{Q}_1\mathtt{Q}_2\cdots\mathtt{Q}_m \quad \mathcal{C}(\mathbf{x}, \mathbf{z})$ for constantly many alternations?

  Can we show that VP = VNP implies that VPH = VP?

Open Questions II

τ**-conjecture for Newton polytopes** [Koiran-Portier-Tavenas-Thomassé]
Suppose $f(x, y)$ is a bivariate polynomial that can be written as $\sum_{i\in[s]}\prod_{j\in[r]} T_{i,j}(x, y)$, where each $T_{i,j}$ has sparsity at most p. Then the Newton polygon of f has poly(s, r, p) vertices.

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Polynomials that can be projected to bivariates in such a way that all of their monomials become vertices of the resulting Newton polygon.

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Can we extend this to $\mathrm{mVP}_{\mathrm{proj}}$?

Thank you!