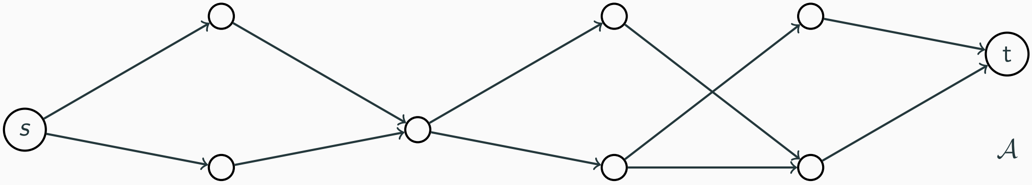
Lower Bounds against Ordered Set-Multilinear ABPs

Prerona Chatterjee [with Deepanshu Kush, Shubhangi Saraf and Amir Shpilka]

September 17, 2024

Algebraic Branching Programs



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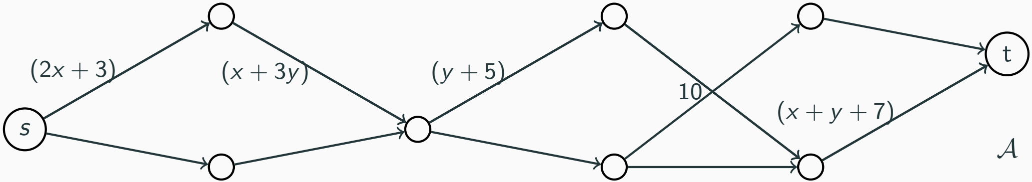

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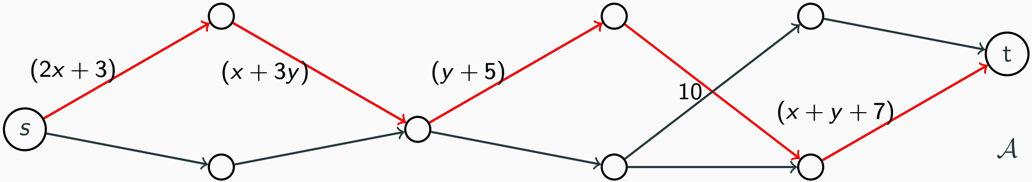

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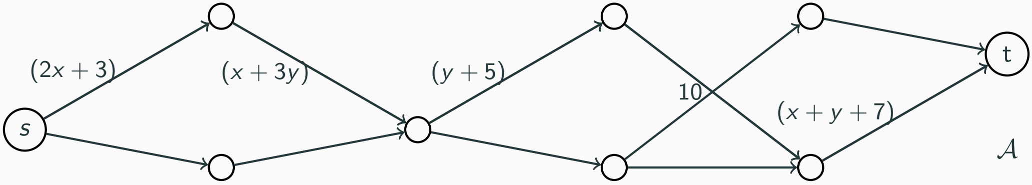

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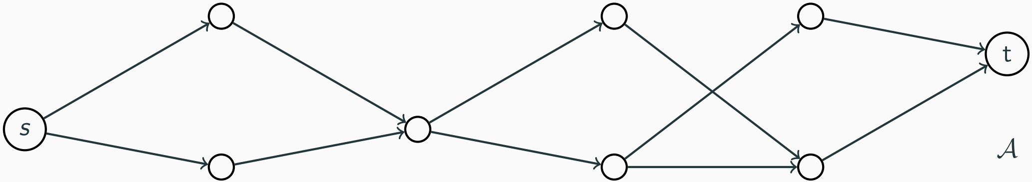

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Question: Find explicit polynomials that can not be computed efficiently by ABPs.

Towards Better ABP Lower Bounds

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[C-Kush-Saraf-Shpilka 24]: For $\omega(\log n) = d \leq n$, there is a polynomial $G_{n,d}(\mathbf{x})$ which is set-multilinear w.r.t $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$, where $|\mathbf{x}_i| \leq n$ for every $i \in [d]$, such that:

- $G_{n,d}$ is computable by a set-multilinear ABP of size $\text{poly}(n)$,
- any $\sum$osmABP computing $G_{n,d}$ must have super-polynomial total-width.

Set-Multilinearity

The variable set is divided into buckets.

$$\mathbf{x} = \mathbf{x}_1 \cup \cdots \cup \mathbf{x}_d \quad \text{where} \quad \mathbf{x}_i = \{x_{i,1}, \ldots x_{i,n_i}\}.$$

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An ABP is set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if every path in it

computes a set-multilinear monomial with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$.

Near Tightness of ABP Set-Multilinearisation

For $\sigma \in S_d$, an ABP is σ-ordered set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

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[C-K-S-S 24]: Super polynomial lower bound against total-width of $\sum$osmABP for a polynomial of degree $d = \omega(\log n)$ that is computable by polynomial-sized ABPs.

Super-Polynomial Lower Bound against $\sum$osmABPs

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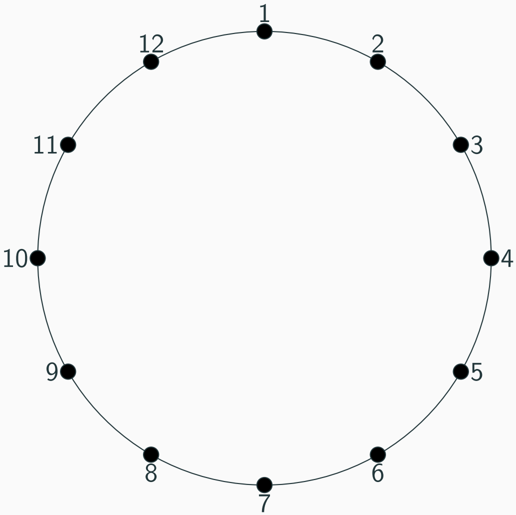
- any $\sum$ osmABP of max-width poly(n) computing $G_{n,d}$ requires total-width $2^{\Omega(d)}$,
- any ordered set-multilinear branching program computing $G_{n,d}$ requires width $n^{\Omega(d)}$.

Proof Ideas

Arc Partition



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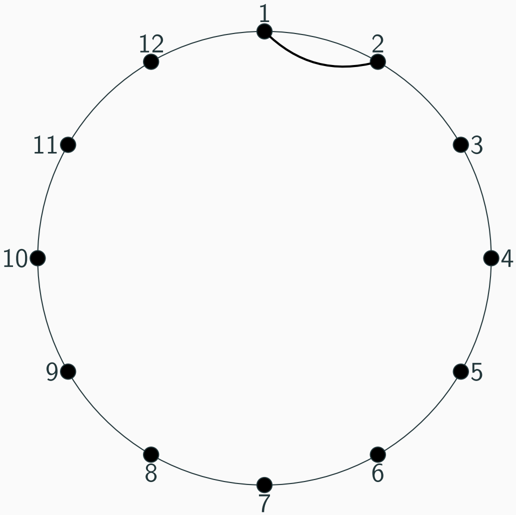

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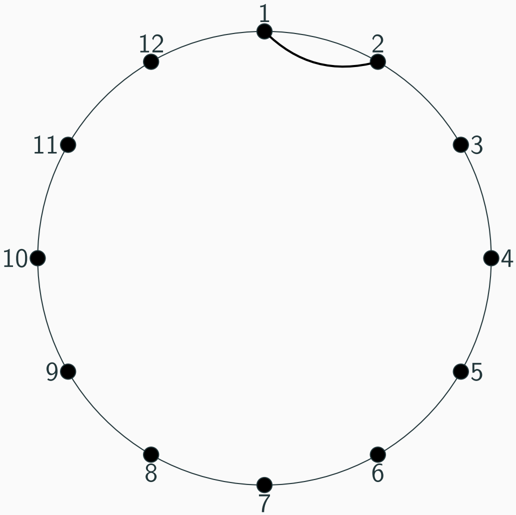

$$\mathcal{P}_1 = \{(1,2)\}$$

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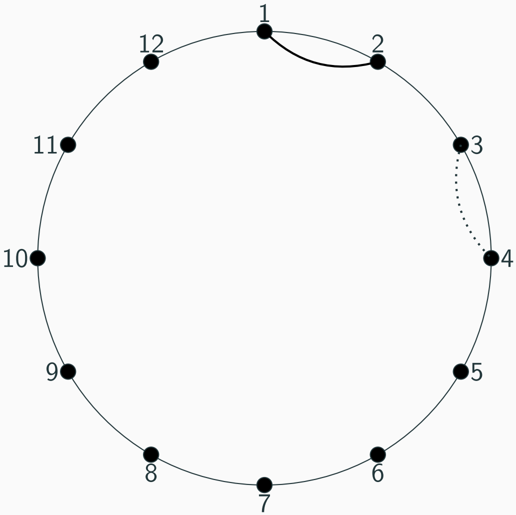

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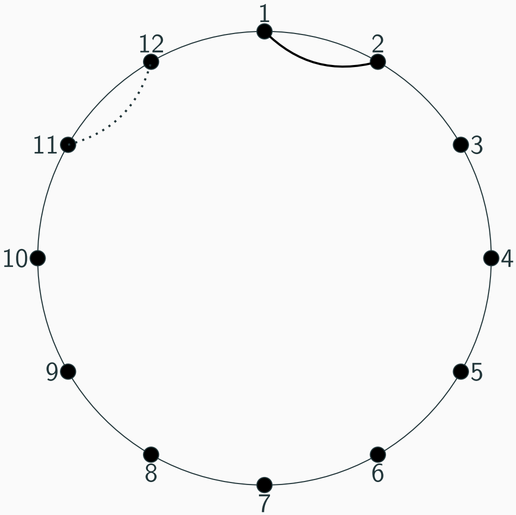

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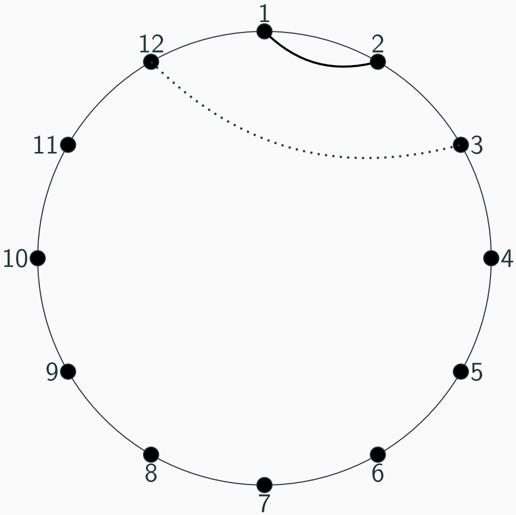

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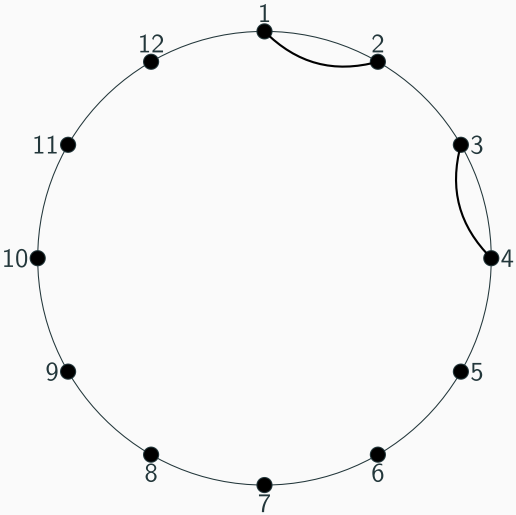

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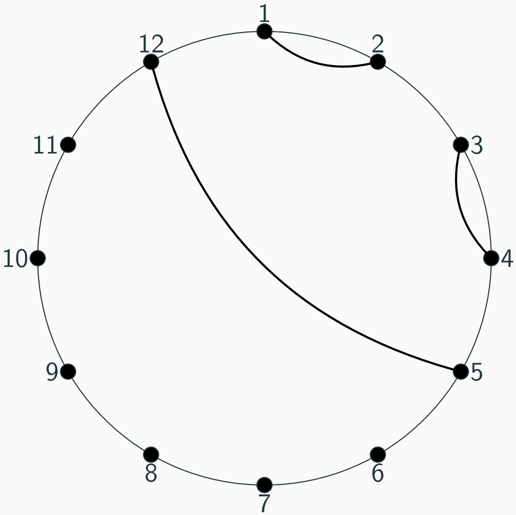

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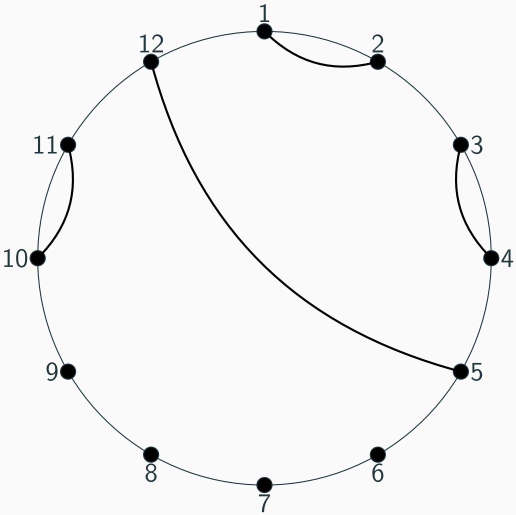

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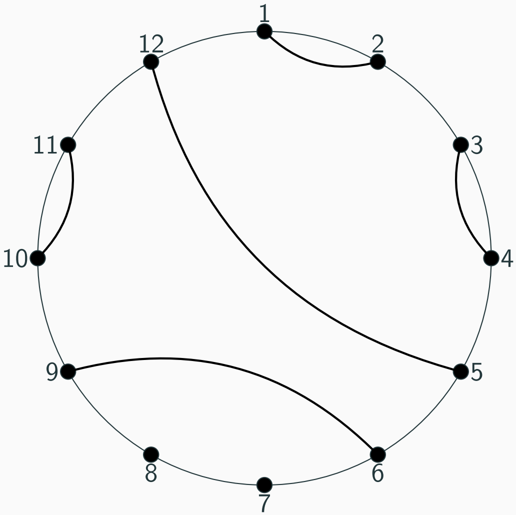

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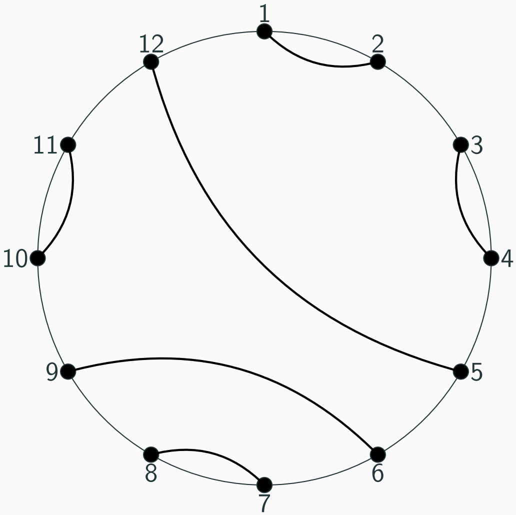

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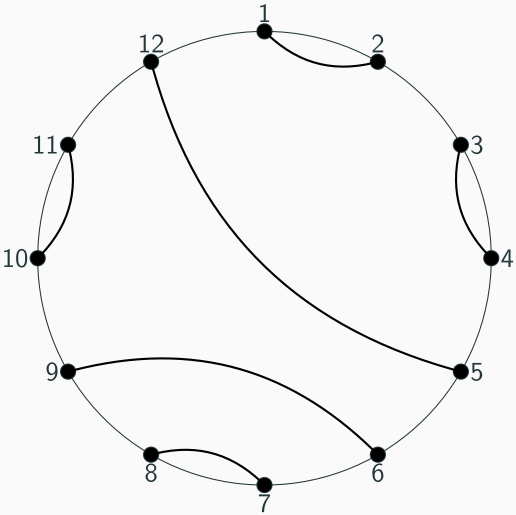

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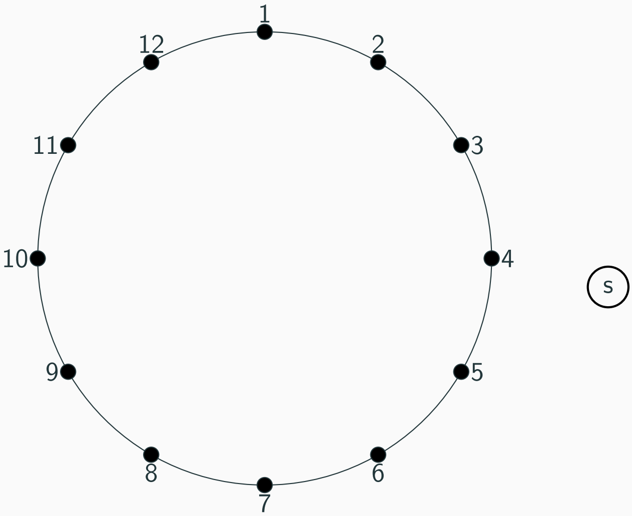
$\mathbf{P}_6 =$ All possibles sequences of such pairs.

The ABP Upper Bound



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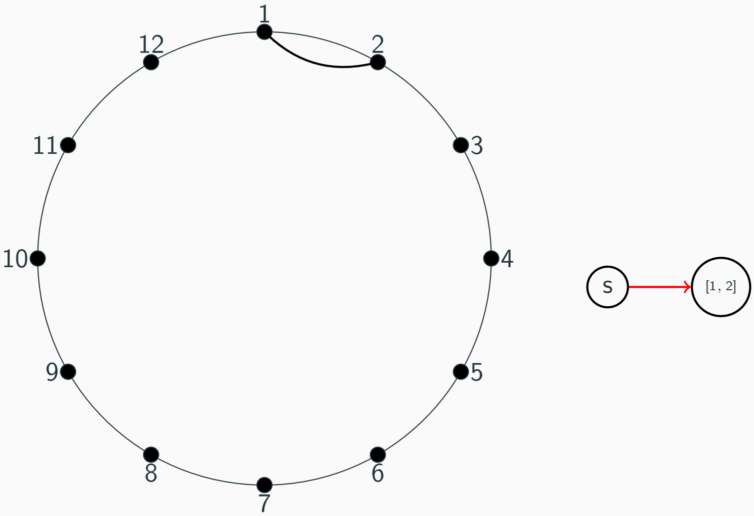

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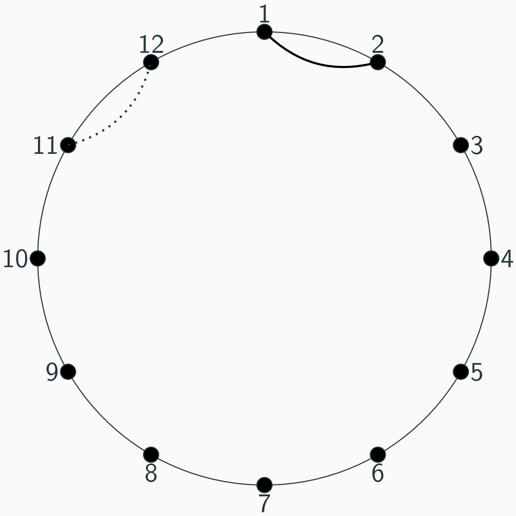

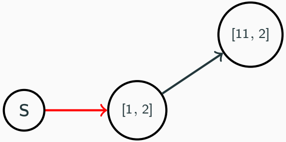

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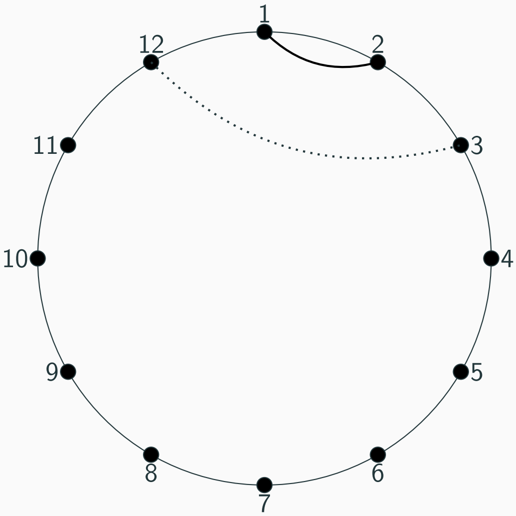

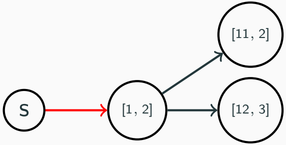

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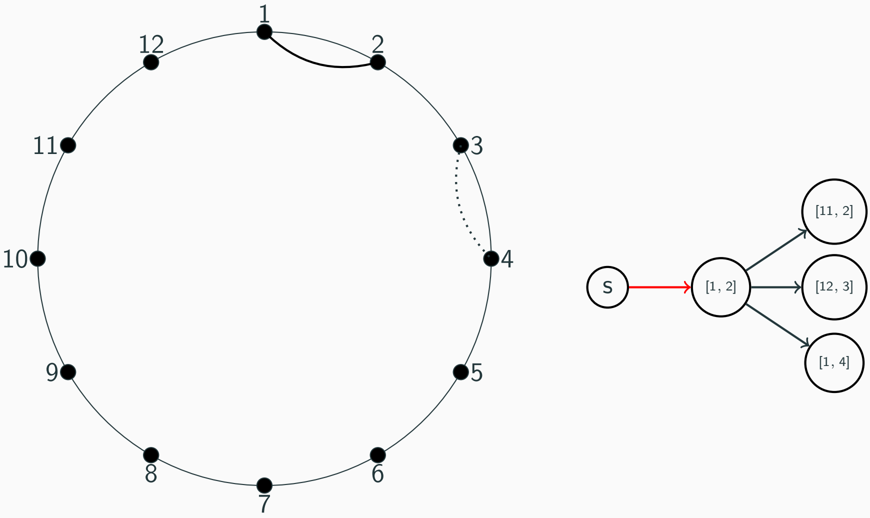

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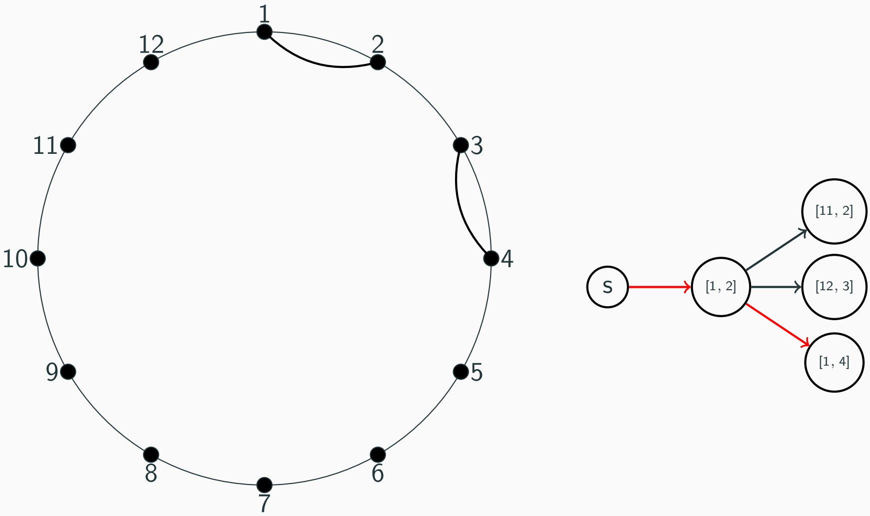

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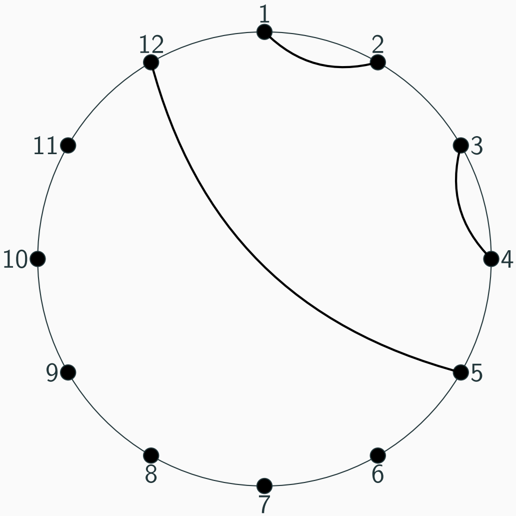

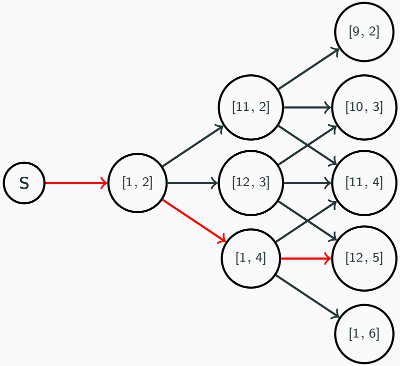

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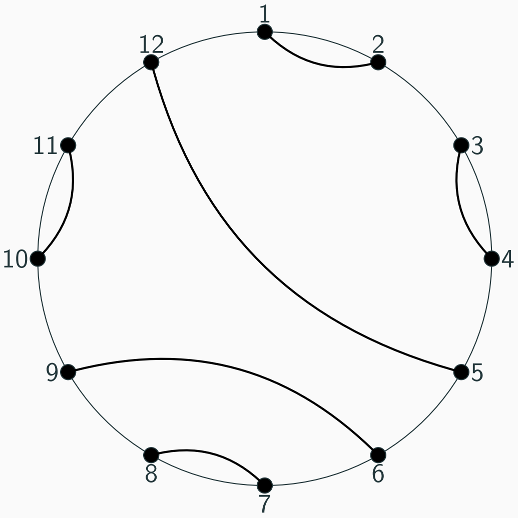

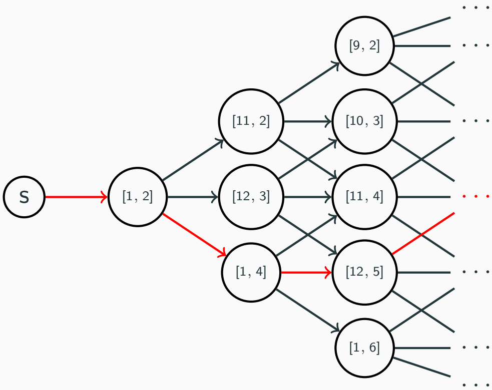

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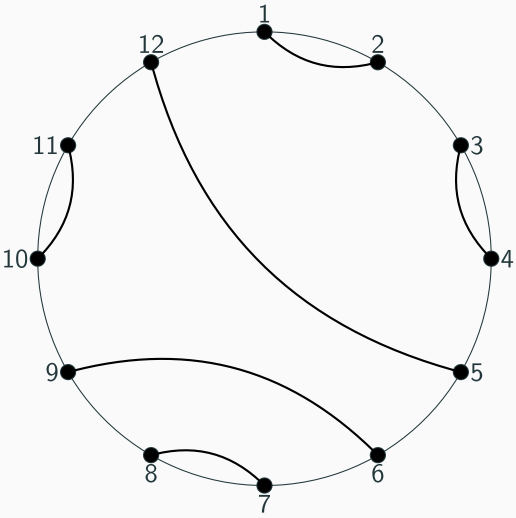
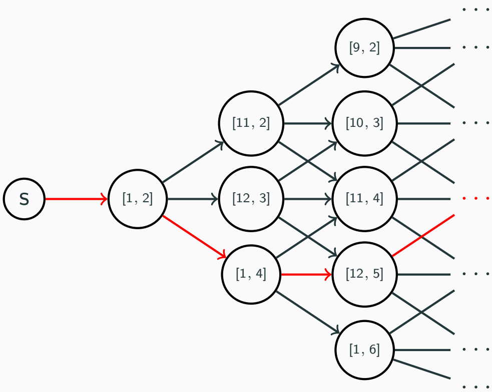


Every path corresponds to an element in $\mathbf{P}_{d/2}$.



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The new pair: $(3, 4)$.

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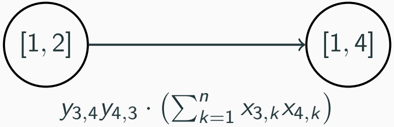

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$y_{3,4}y_{4,3} \cdot \left(\sum_{k=1}^{n} x_{3,k}x_{4,k}\right)$

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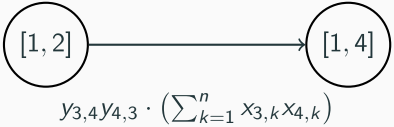
$(y_{3,4}y_{4,3})$: To select.



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$$
\begin{array}{c|ccccc|}
 & x_{4,1} & x_{4,2} & \cdots & \cdots & x_{4,n} \\
\hline
x_{3,1} & 1 & 0 & \cdots & \cdots & 0 \\
x_{3,2} & 0 & 1 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & & & \vdots \\
\vdots & \vdots & \vdots & & & \vdots \\
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\hline
\end{array}
$$

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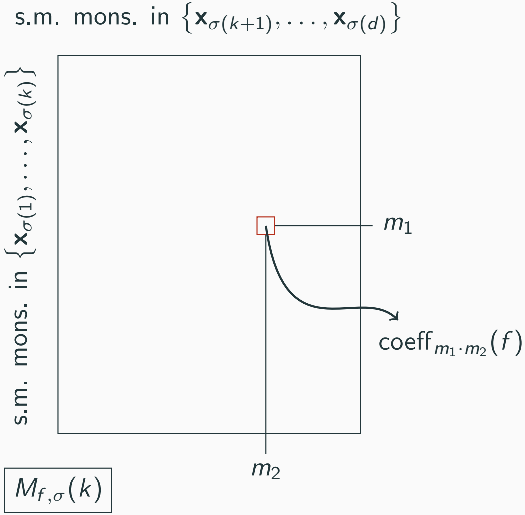

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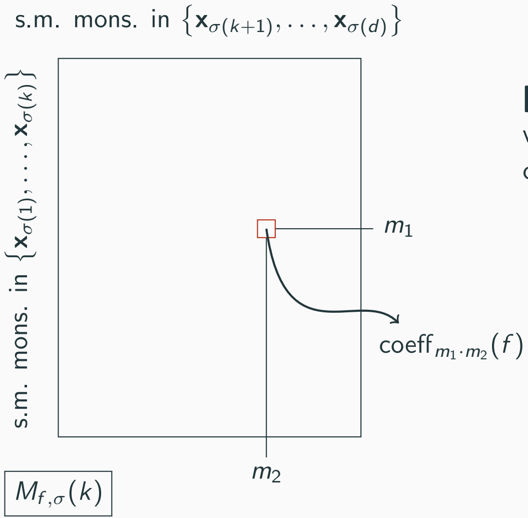
Lower Bound for a single osmABP



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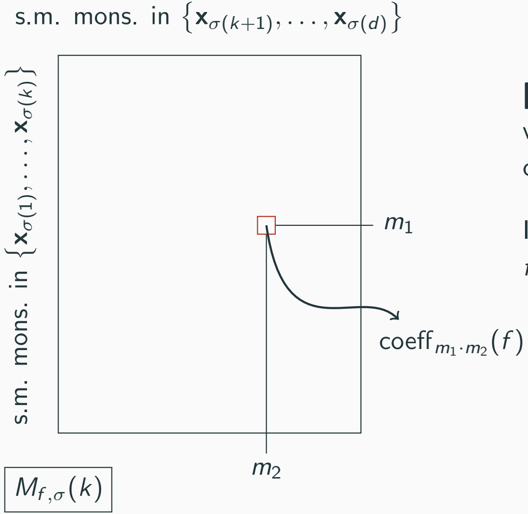

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If $\mathcal{A}$ is the smallest osmABP (in order σ) computing f, then

$$\mathrm{size}(\mathcal{A}) = \sum_{i=1}^{d} \mathrm{rank}(M_{f,\sigma}(k)).$$

Lower Bound for a single osmABP (contd.)

$$G_{n,d} = \sum_{\mathcal{P} \in \mathbf{P}_{d/2}} \prod_{(i,j) \in \mathcal{P}} y_{i,j} y_{j,i} \cdot \left(\sum_{k=1}^{n} x_{i,k} x_{j,k} \right).$$

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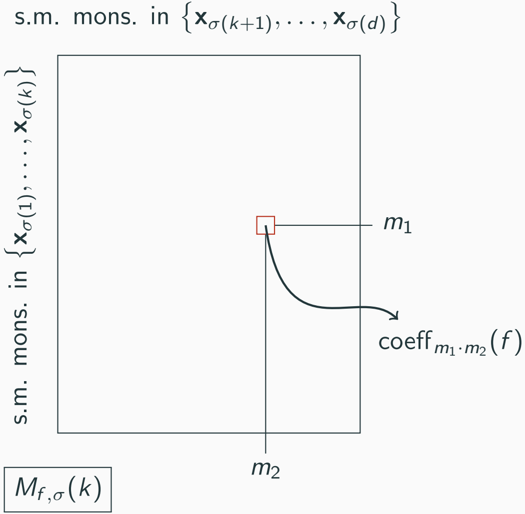

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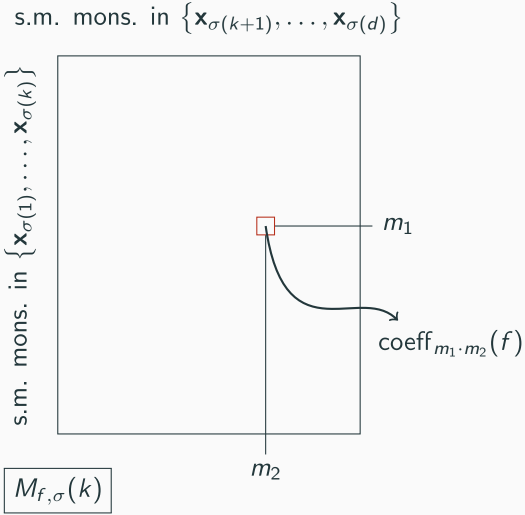

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  Therefore,

$$\text{rank}(M_{G_{n,d},\sigma}(d/2)) = \Omega(n^{d/8}).$$

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Thank you!!!

Discussion

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- $n^{\Omega(\log n)}$ set-multilinear formula LB for $\mathrm{IMM}_{n,n}$ implies formula LB due to self-reducibility.

[Kush-Saraf]

The Result: There is an n-variate polynomial, $G_{n,n}$, of degree n computable by a set-multilinear ABP of width $s = \text{poly}(n)$ such that any set-multilinear formula computing it has size $n^{\Omega(\log n)}$.

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How different are the powers of smABP**s and** osmABP**s?**

What Our Results Say

- Optimal separation between smABP and osmABP.
- Exponential separation between smABP and $\sum$ osmABP when $d = \Theta(n)$.
- Super-polynomial separation between smABP and $\sum$ osmABP when $d = \omega(n)$.

Thank you again!!!