Lower Bounds Against Sums of Ordered Set-Multilinear ABPs

Prerona Chatterjee [with Deepanshu Kush (UoT), Shubhangi Saraf (UoT), Amir Shpilka (TAU)]
Tel Aviv University

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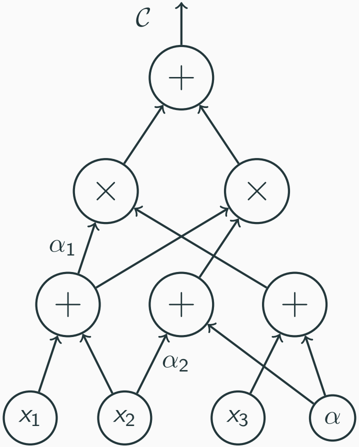
Algebraic Models of Computation



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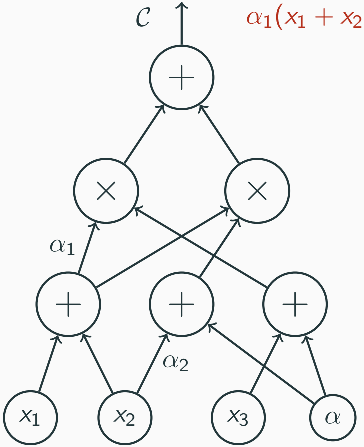

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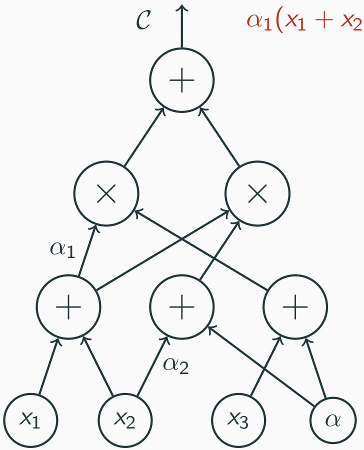

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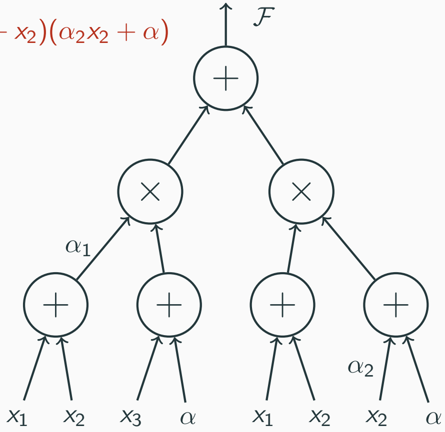
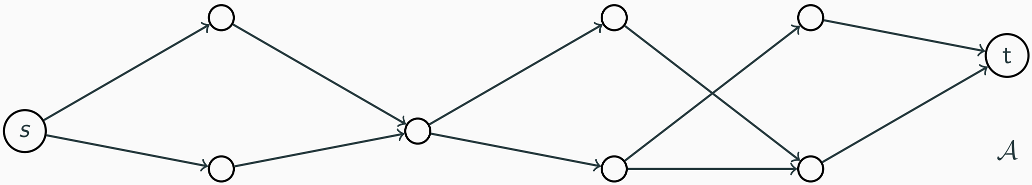


Algebraic Branching Programs



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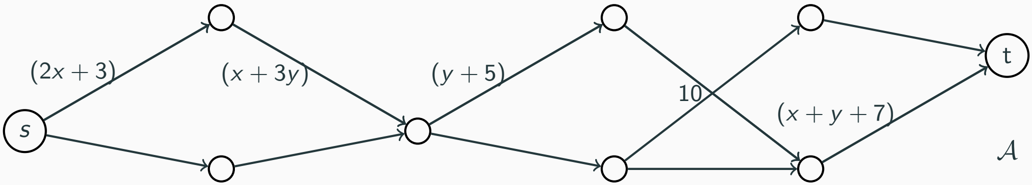

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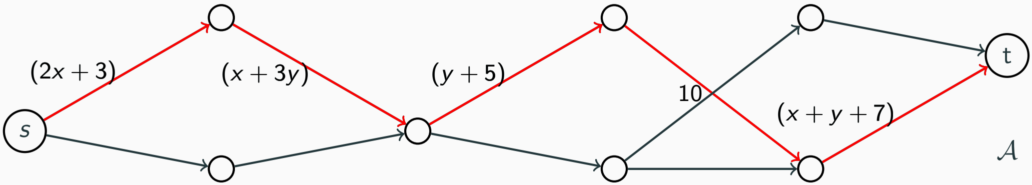

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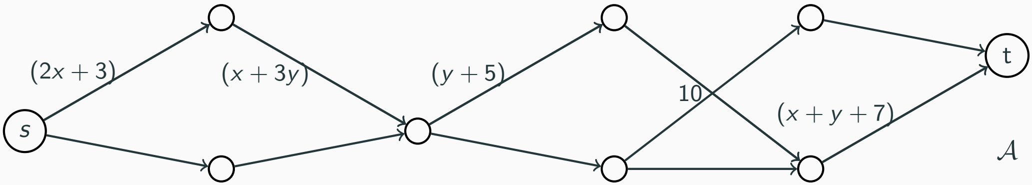

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- Polynomial computed by the ABP: $f_{\mathcal{A}}(\mathbf{x}) = \sum_p \mathrm{wt}(p)$

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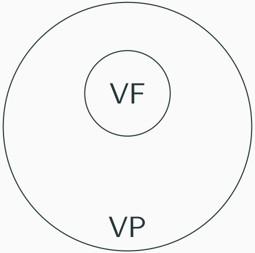

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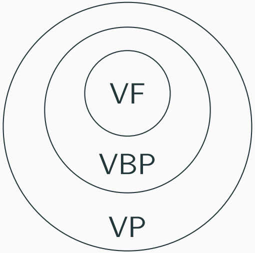

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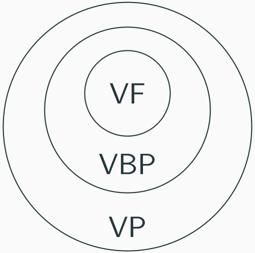

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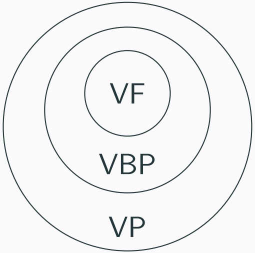

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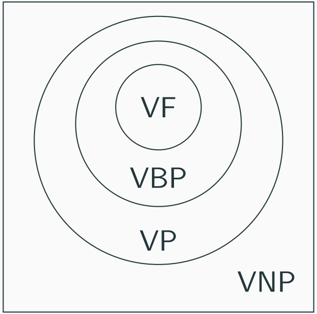
VNP: Explicit Polynomials



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Our Results

Set-Multilinearity

The variable set is divided into buckets.

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$$\mathrm{IMM}_{n,d} = \sum_{1 \le i_1, \ldots, i_{d-1} \le n} x^{(1)}_{1,i_1} \cdot \left(\prod_{j=2}^{d-1} x^{(j)}_{i_{j-1},i_j} \right) \cdot x^{(d)}_{i_{d-1},i_d}.$$

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Note: $\mathrm{IMM}_{n,d}$ is complete for the set of polynomials of degree d that are computable by osmABPs (also smABPs and ABPs) which had width at most n in every layer.

[Tavenas-Limaye-Srinivasan]

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- $n^{\Omega(\log n)}$ set-multilinear formula LB for $\mathrm{IMM}_{n,n}$ implies formula LB due to self-reducibility.

[Kush-Saraf]

The Result: There is an n-variate polynomial, $G_{n,n}$, of degree n computable by a set-multilinear ABP of size $s = \text{poly}(n)$ such that any set-multilinear formula computing it has size $n^{\Omega(\log n)}$.

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How different are the powers of smABP**s and** osmABP**s?**

Our Results

Our First Result: There is a polynomial $G_{n,d}(\mathbf{x})$ which is set-multilinear with respect to $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$, where $|\mathbf{x}_i| \leq n$ for each $i \in [d]$, such that:

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Our Second Result: There is polynomial $G_{n,n}(\mathbf{x})$ which is set-multilinear with respect to a set of $\Theta(n)$ buckets, each of size $\Theta(n)$, such that

- it has a set-multilinear branching program of size poly(n),
- but any $\sum$osmABP computing $G_{n,n}(\mathbf{x})$ requires total-width $\exp(\Omega(n^{1/1000}))$.

The [Bhargav-Dwivedi-Saxena] Perspective

The Result: Let $P_{n,d}(\mathbf{x})$ be a polynomial of degree d that is set-multilinear w.r.t the partition $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ where $|\mathbf{x}_i| \leq n$ for every $i \in [d]$, If $P_{n,d}$ can be computed by an ABP of size s, then it can also be computed by a $\sum$osmABP of max-width s and total-width $2^{O(d \log d)} s$.

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Our Third Result: For $\omega(\log n) = d \leq n$, there is a polynomial $G_{n,d}(\mathbf{x})$ which is set-multilinear w.r.t $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$, where $|\mathbf{x}_i| \leq n$ for every $i \in [d]$, such that:

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Results for polynomials in VP, VNP

Our Fourth Result: There is polynomial family $\{F_{n,n}(\mathbf{x})\}$, in VP, which is set-multilinear with respect to a set of $\Theta(n)$ buckets, each of size $\Theta(n)$, such that

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Related Results

[Bhargav-Dwivedi-Saxena]

Any $\sum$osmABP computing $\mathrm{IMM}_{n,n}$ which has max-width $n^{o(1)}$ must have $2^{\Omega(n)}$ summands.

Related Results

[Bhargav-Dwivedi-Saxena]

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[Arvind-Raja]

Any $\sum_{i=1}^{t}$ osmABP computing the $n \times n$ permanent polynomial has max-width $2^{\Omega(n/t)}$.

Sum of ROABPs

ROABPs: An ABP computing $f \in \mathbb{F}[x_1, \ldots x_n]$ is an ROABP, in order $\sigma \in S_n$, if there are n layers in the ABP and, for each $i \in [n]$, every edge in layer i is labelled a polynomial in $x_{\sigma(i)}$.

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- Suppose g_f is computable by a $\sum$ ROABP, $\sum \mathcal{A}_j$, with $\mathcal{A}_j$ ordered σ_j.

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- Define $g_f(x_1, \ldots, x_d) = \sum_{\mathbf{e} \in [n]^d} \prod_{i=1}^{e_i} x_i^{e_i} \cdot \mathrm{coeff}_{x_{i,e_i}}(f)$.
- Suppose g_f is computable by a $\sum$ROABP, $\sum \mathcal{A}_j$, with $\mathcal{A}_j$ ordered σ_j.
- Construct a $\sum$osmABP, $\sum \mathcal{A}'_j$, with $\mathcal{A}'_j$ ordered σ_j, by replacing $x_i^{e_i}$ by x_{i,e_i} for $e_i \in [n_i]$ and erasing other components on each edge.

Sum of ROABPs

ROABPs: An ABP computing $f \in \mathbb{F}[x_1, \ldots x_n]$ is an ROABP, in order $\sigma \in S_n$, if there are n layers in the ABP and, for each $i \in [n]$, every edge in layer i is labelled a polynomial in $x_{\sigma(i)}$.

Note: $\sum$ osmABP LBs imply $\sum$ ROABP LBs.

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- Check that this $\sum$ osmABP computes f and use known LB.

[Ramya-Rao]

There exists an explicit multilinear polynomial family $\{g_n(x_1, \ldots, x_n)\}$, in VP, such that any $\sum$ ROABP computing g_n has total width $2^{\Omega\left(\frac{n^{1/6}}{\log n}\right)}$.

Related Results

[Ramya-Rao]

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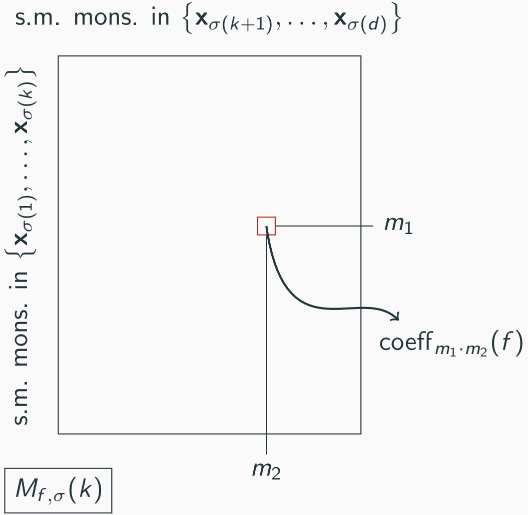
[Ghoshal-Rao]

There exists an explicit multilinear polynomial family $\{g_n(x_1, \ldots, x_n)\}$, in VBP, such that any $\sum$ ROABP computing g_n that has max-width poly(n) must have total width $2^{\Omega\left(n^{1/500}\right)}$.

Proof Overviews

Lower Bound for osmABPs



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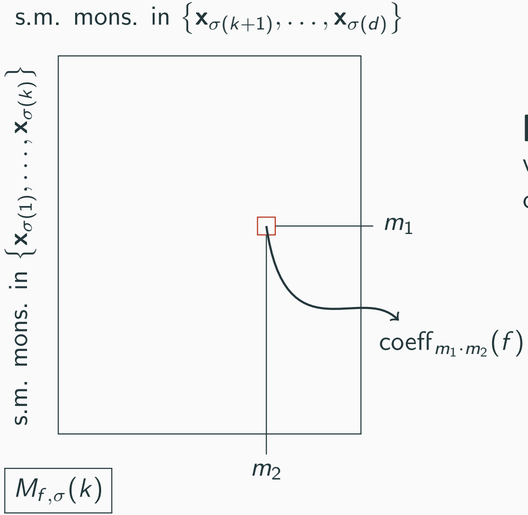

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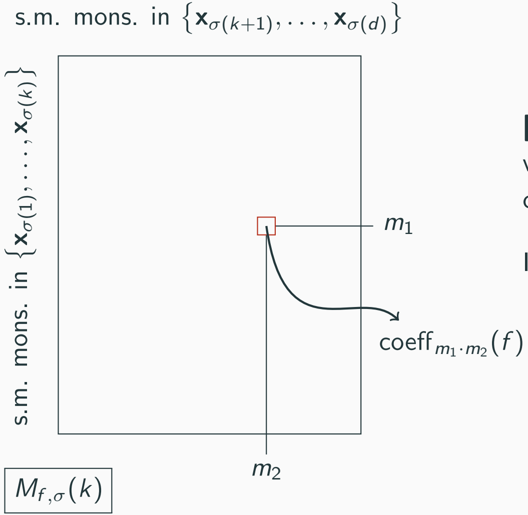

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If $\mathcal{A}$ is the smallest osmABP computing f, then

$$\mathrm{size}(\mathcal{A}) = \sum_{i=1}^{d} \mathrm{rank}(M_{f,\sigma}(k)).$$

Arc Partition

Fix an even $d \in \mathbb{N}$ arbitrarily. We will define a distribution $\mathcal{D}$ over the set $\{-k, +k\}^d$.

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The Hard Polynomial

$$G_{n,d} = \sum_{\mathcal{P}_{d/2}} \prod_{(i,j)\in\mathcal{P}} y_{i,j} y_{j,i} \cdot \left(\sum_{k=1}^{n} x_{i,k} x_{j,k} \right).$$

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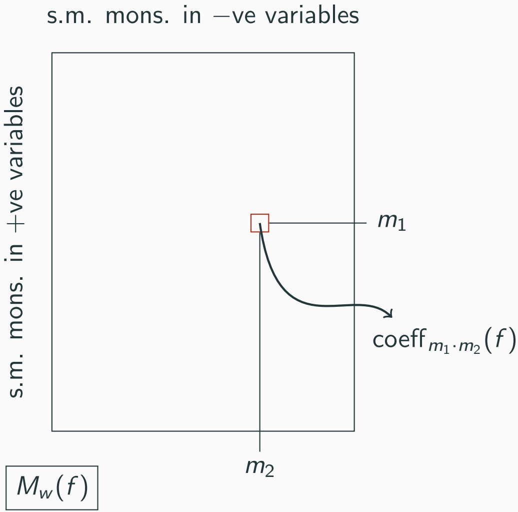

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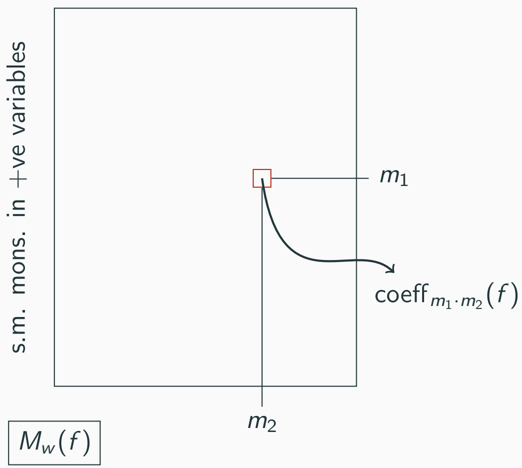

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[Bhargav-Dwivedi-Saxena]

The Result: Let $P_{n,d}(\mathbf{x})$ be a polynomial of degree d that is set-multilinear w.r.t. the partition $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ where $|\mathbf{x}_i| \leq n$ for every $i \in [d]$, If $P_{n,d}$ can be computed by an ABP of size s, then it can also be computed by a $\sum$osmABP of max-width s and total-width $2^{O(d \log d)}s$.

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$$f = \sum_{i=1}^{t} g_i \quad \text{where} \quad g_i = \sum_{u_1,\ldots,u_{q-1}} \prod_{j=1}^{q} g^{(i)}_{u_{j-1},u_j}.$$

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- Using union bound we will get the lower bound if we can show that:

When $w \sim \mathcal{D}$, for $g = \sum_{u_1,\ldots,u_{q-1}} \prod_{j=1}^{q} g_{u_{j-1},u_j}$, with high probability $\mu_w(g)$ is far from full.

Our Proof Strategy (contd.)

To show: For $g = \sum_{u_1,\ldots,u_{q-1}} \prod_{j=1}^{q} g_{u_{j-1},u_j}$, $\mu_w(g)$ is far from full with high probability if $w \sim \mathcal{D}$.

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- For hard polynomial in VP, choose q wisely and then use Chernoff bound.
- For hard polynomial in VBP, choose q wisely and then use *many violations lemma*.

Open Questions

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1. PIT for $\sum$ ROABP?

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2. Super-quadratic lower bounds against smABPs?

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Thank you!