Lower Bounds for some Algebraic Models of Computation

Prerona Chatterjee

April 2, 2024

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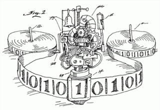
Given a boolean function f on n inputs, how many steps are required by a Turing machine to compute the f (in terms of n)?



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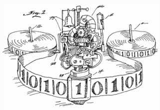

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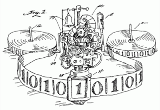

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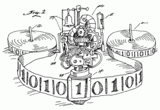

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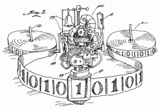

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Circuit Complexity | **Communication Complexity** | **Quantum Complexity**

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Matrix Multiplication Exponent (ω): Smallest number k such that the product of two $n \times n$ matrices can be found using n^k multiplications.

Other Problems that I have worked on

Algebraic Independence Testing: Given polynomials $f_1, \ldots, f_m \in \mathbb{F}[x_1, \ldots, x_n]$, check if there exists $0 \not\equiv A \in \mathbb{F}[x_1, \ldots, x_n]$ such that $A(f_1, \ldots, f_n) \equiv 0$.

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Parametric Shortest Paths: Variants of the shortest path problem when the edge weights are labelled with polynomials.

Results in restricted setting with Gajjar, Radhakrishnan, Varma: **[GVCR 21]**.

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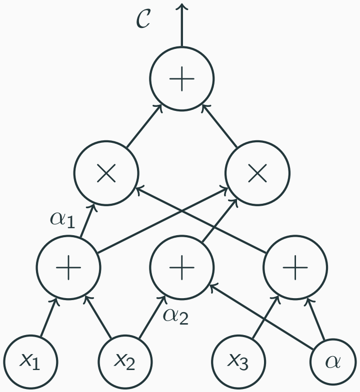

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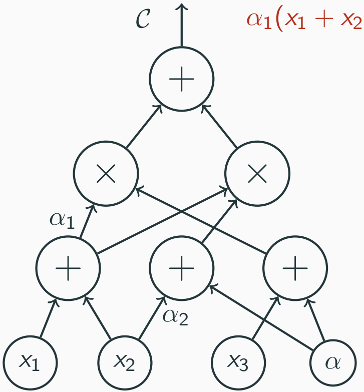

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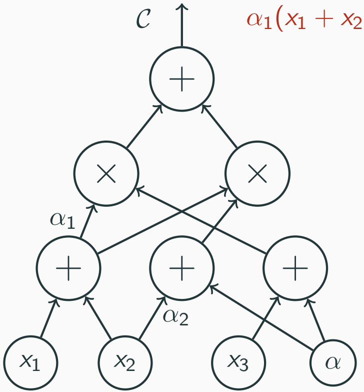

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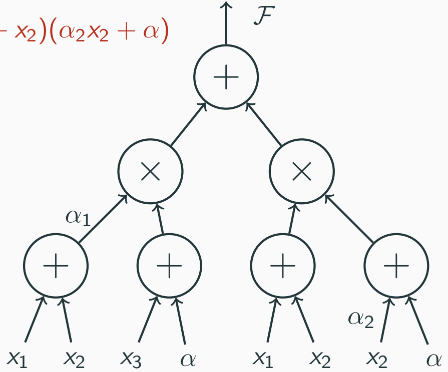

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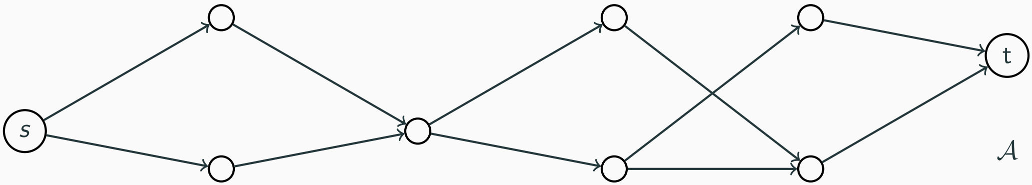

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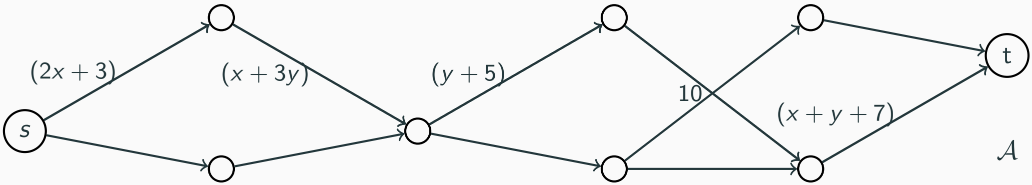

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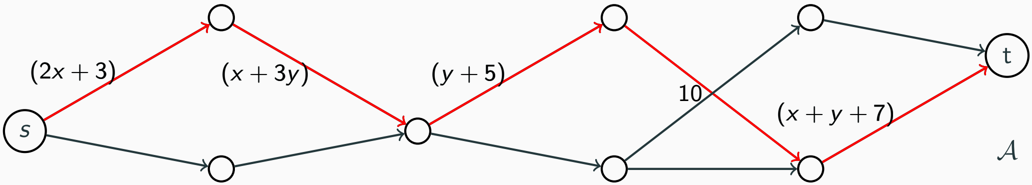

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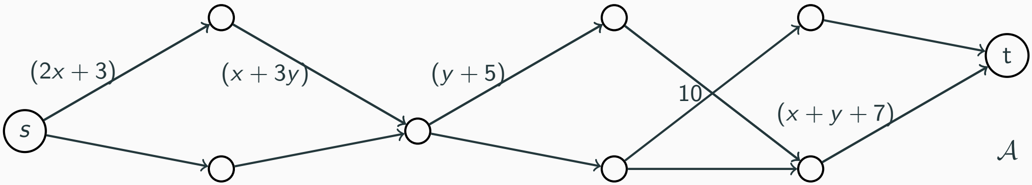

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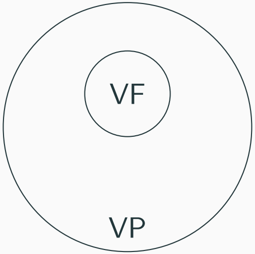

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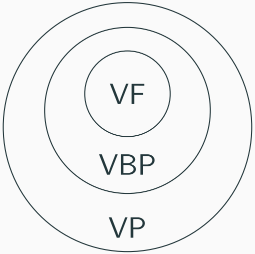

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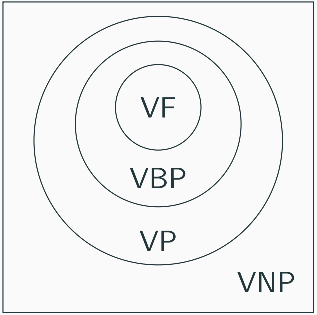

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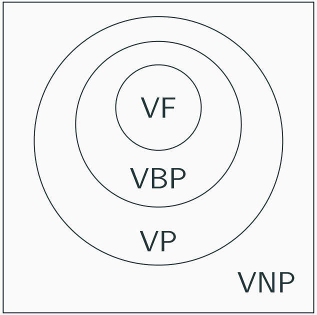

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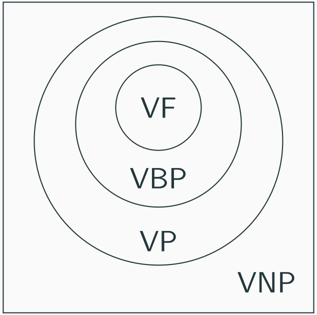

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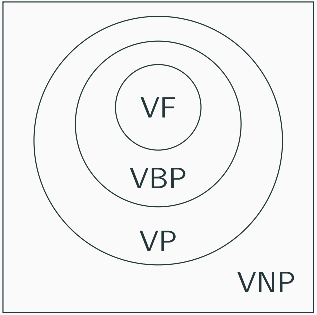

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Other Motivating Questions: Are the other inclusions tight?

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[C-Kumar-She-Volk 22]: Any formula computing $\mathrm{ESYM}_{n,0.1n}(\mathbf{x})$ requires $\Omega(n^2)$ vertices.

$$\mathrm{ESYM}_{n,d}(\mathbf{x}) = \sum_{i_1 < \cdots < i_d \in [n]} x_{i_1} \cdots x_{i_d}.$$

How does one make progress?

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Show that if a structured n-variate, degree-d polynomial is computable by a general model of size s, then they can also be computed by a structured model of size $\mathrm{func}(s, n, d)$ for some function func.

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The lower bound is $n^{\Omega(\sqrt{d})}$ for depth-3 and depth-4.

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[C-Kush-Saraf-Shpilka 24]: For $\omega(\log n) = d \leq n$, there is a polynomial $G_{n,d}(\mathbf{x})$ which is set-multilinear w.r.t $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$, where $|\mathbf{x}_i| \leq n$ for every $i \in [d]$, such that:

- $G_{n,d}$ is computable by a set-multilinear ABP of size $\text{poly}(n)$,
- any $\sum$osmABP computing $G_{n,d}$ must have super-polynomial total-width.

Set-Multilinearity

The variable set is divided into buckets.

$$\mathbf{x} = \mathbf{x}_1 \cup \cdots \cup \mathbf{x}_d \quad \text{where} \quad \mathbf{x}_i = \{x_{i,1}, \ldots x_{i,n_i}\}.$$

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f is set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

every monomial in f has exactly one variable from $\mathbf{x}_i$ for each $i \in [d]$.

Set-Multilinearity

The variable set is divided into buckets.

$$\mathbf{x} = \mathbf{x}_1 \cup \cdots \cup \mathbf{x}_d \quad \text{where} \quad \mathbf{x}_i = \{x_{i,1}, \ldots x_{i,n_i}\}.$$

f is set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

every monomial in f has exactly one variable from $\mathbf{x}_i$ for each $i \in [d]$.

An ABP is set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if every path in it

computes a set-multilinear monomial with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$.

Near Tightness of ABP Set-Multilinearisation

For $\sigma \in S_d$, an ABP is σ-ordered set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

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Further, there is a non-commutative circuit of size $O(n \log^2 n)$ that computes $\mathrm{OSym}_{n,n/2}(\mathbf{x})$.

ABP vs Formula in the Non-Commutative Setting

[Nisan 91]: Any ABP computing $\mathrm{Pal}_n(x_0, x_1) = \sum_{w \in \{0,1\}^{n/2}} \mathbf{x}_w \cdot \mathbf{x}_{w^R}$ has size $2^{\Omega(n)}$.

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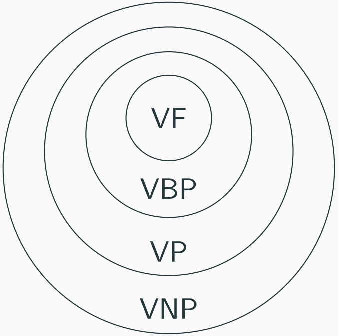
If an n-variate polynomial is abecedarian with respect to $\{X_1, \ldots, X_m\}$ for $m = \log n$, then any formula computing f can be made abecedarian with only $\text{poly}(n)$ blow-up in size.

Classes Beyond VNP



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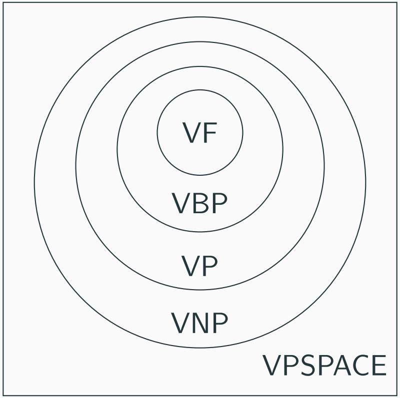
$\mathsf{VPSPACE}_b$: Polynomials whose coefficients can be computed in PSPACE/ poly and have degree bounded by poly(n).



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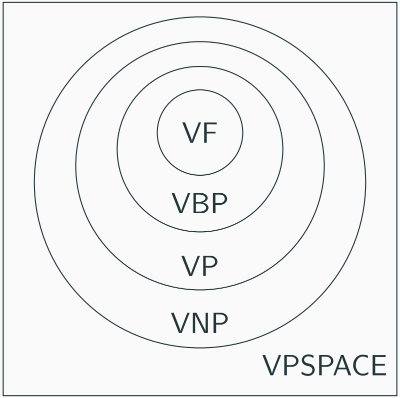

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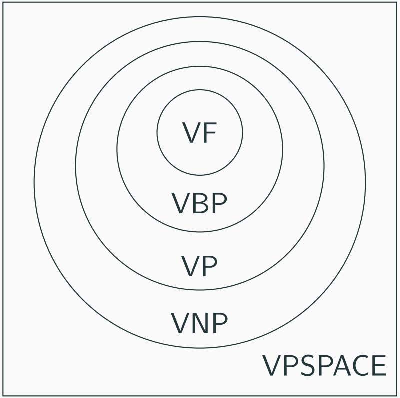

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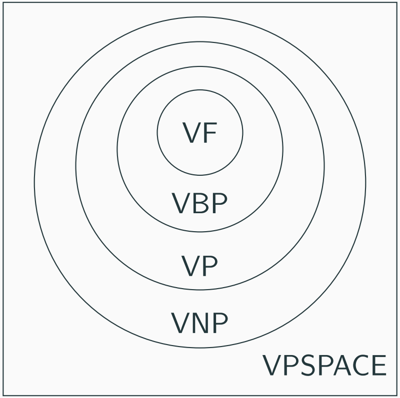
[C-Gajjar-Tengse 23]: $\mathsf{VNP} \neq \mathsf{VPSPACE}_b$ in the monotone setting.



Some Proof Ideas

Super-Polynomial Lower Bound against $\sum$osmABPs

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- any ordered set-multilinear branching program computing $G_{n,d}$ requires width $n^{\Omega(d)}$.

The Hard Polynomial

$y_{1,2}y_{2,1} \cdot \left(\sum_{k=1}^{n} x_{1,k}x_{2,k}\right)$

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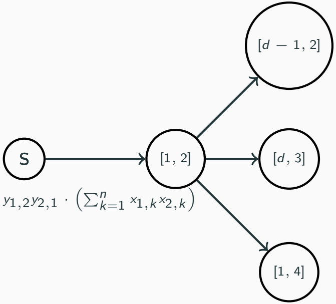

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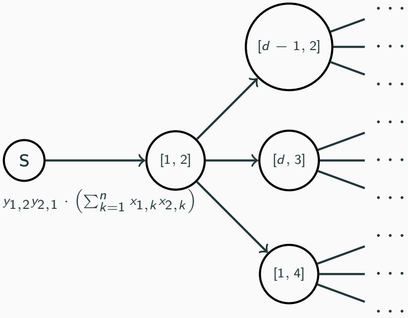

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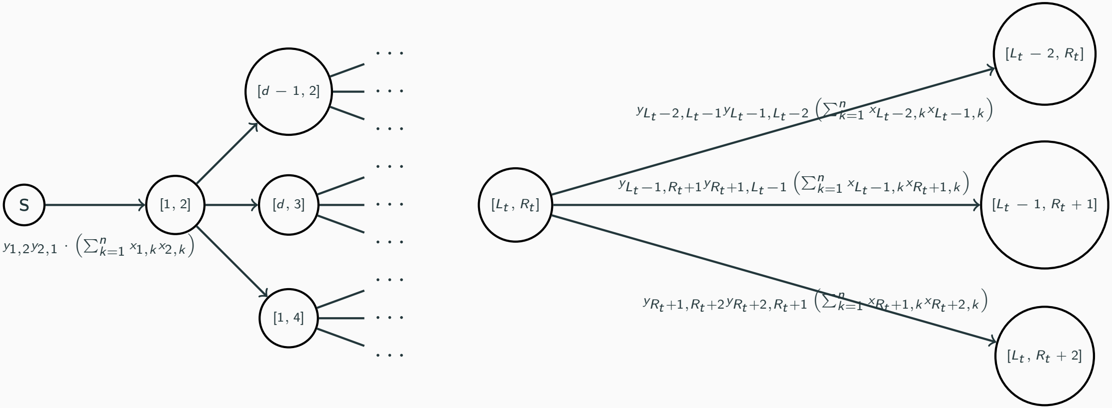

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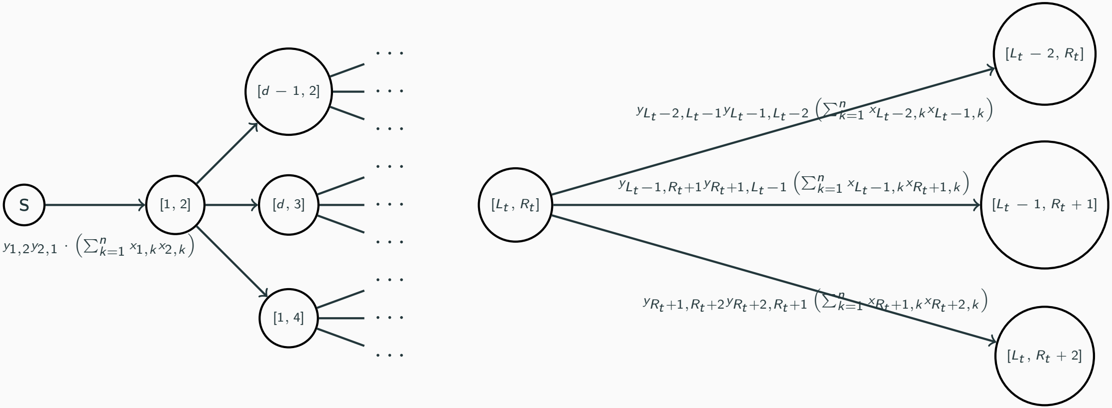

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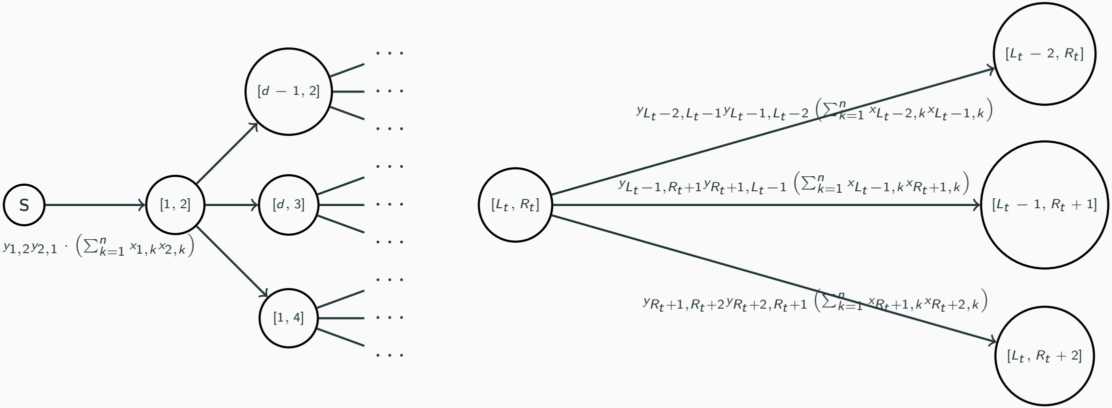

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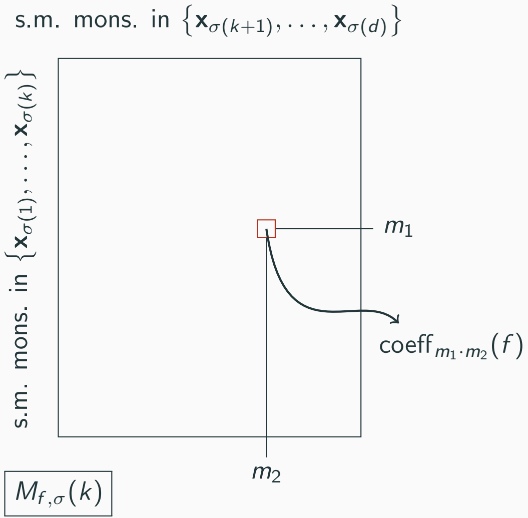
Lower Bound for a single osmABP



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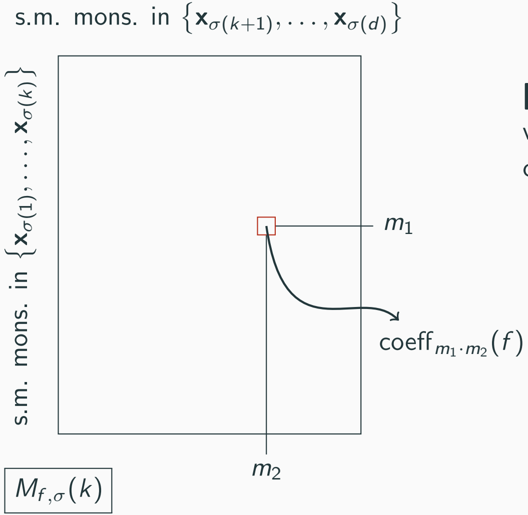

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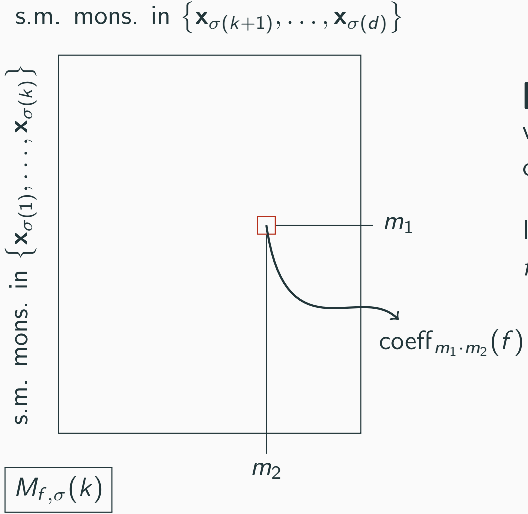

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If $\mathcal{A}$ is the smallest osmABP (in order σ) computing f, then

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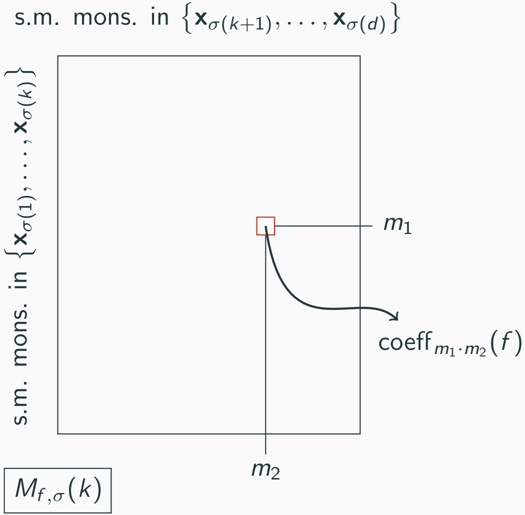

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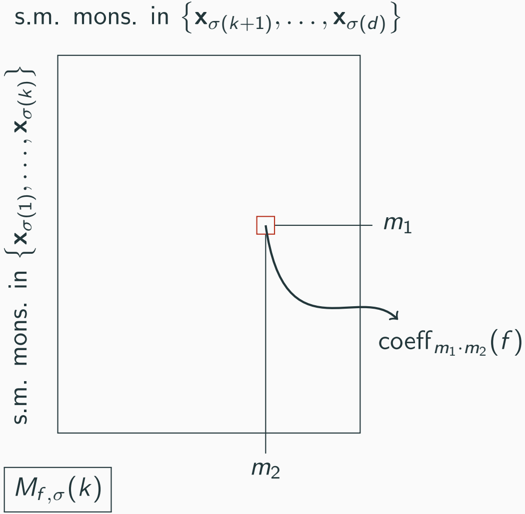

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  Therefore,

$$\mathrm{rank}(M_{G_{n,d},\sigma}(d/2)) = \Omega(n^{d/8}).$$

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[Carmosino-Impagliazzo-Lovett-Mihajlin 18]

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Main Observation: If $f_1, \ldots, f_k$ are simultaneously computable by a homogeneous non-commutative circuit of size s,

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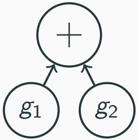

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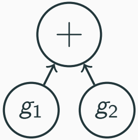

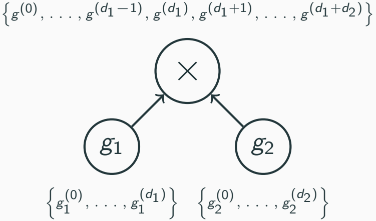

Open Questions in Algebraic Complexity

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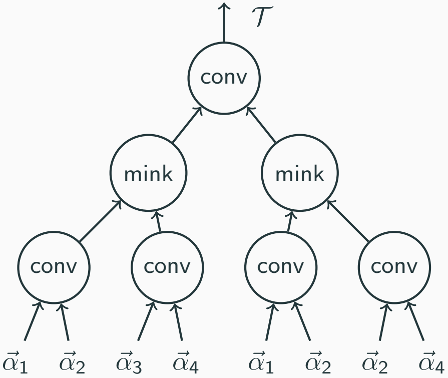
Branching Out

Upper Bounding Vertices in Some Structured Polytopes



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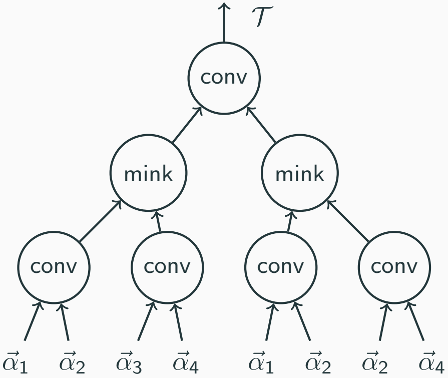

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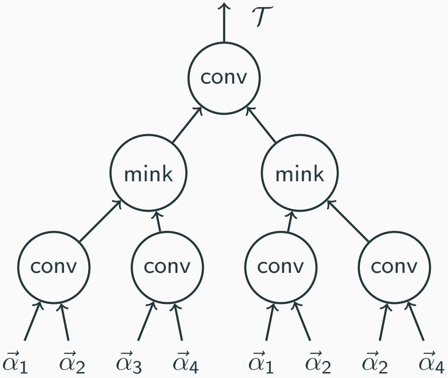

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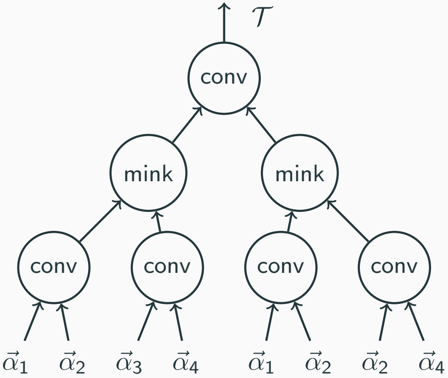
Corollary: Let G be a directed graph on n vertices with two special vertices s and t, and edge weights of the form

$$w_e(\lambda_1, \lambda_2, \ldots, \lambda_d) = a_{1,e}\lambda_1 + a_{2,e}\lambda_2 + \ldots + a_{d,e}\lambda_d.$$

Then the number of different shortest s–t paths in G (as $\lambda_1, \ldots, \lambda_d$ varies) is at most $n^{4(\log n)^{d-1}}$.

Upper Bounding Vertices in Some Structured Polytopes



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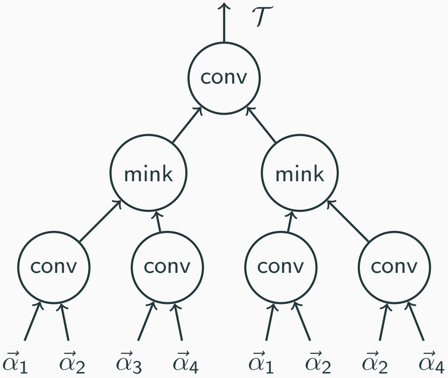

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Open for $d \geq 3$.

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  - $\mathcal{C}$ has a hazard at $(1, 1, u)$.
- Let $\mathcal{C}' \equiv (x \wedge (y \vee z)) \vee (y \wedge \neg z)$.
  - $\mathcal{C}'$ is hazard-free.

Teaching

Courses I would be happy to teach

Basic Courses

- Discrete Structures
- Automata Theory
- Data Structures and Algorithms
- Theory of Computation
- Algorithms and Complexity
- Automata Theory and Logic
- Computer Programming
- Formal Methods in CS
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I would be happy to teach/design other courses depending on interest and/or requirement.