Lower Bounds Against Sums of Ordered Set-Multilinear ABPs

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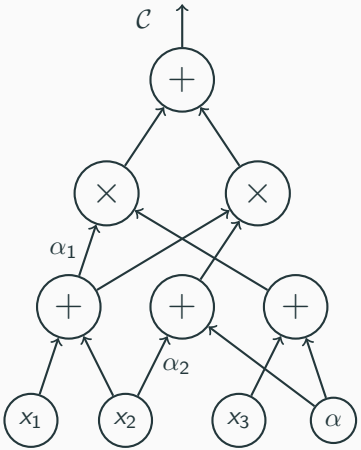
Theorem [Shamir 79, Lipton 94]: If $h(x) = \prod_{i=1}^{d}(x - i)$ can be computed using poly$(\log d)$ additions and multiplications, then integer factoring is in P/poly.

Algebraic Circuit Complexity



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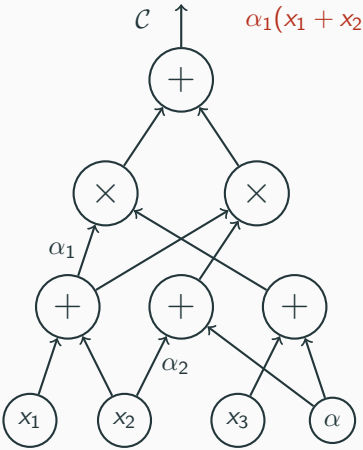

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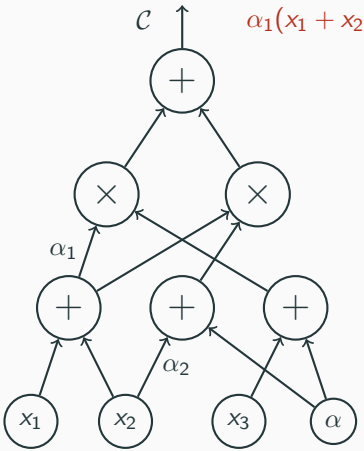

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Polynomials over n variables of degree d.

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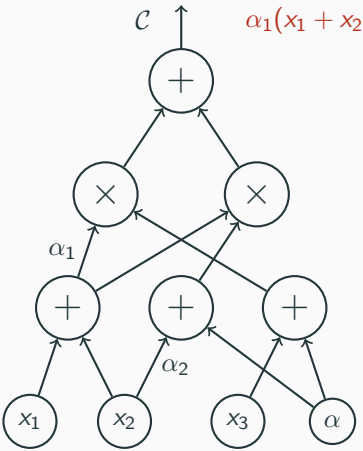

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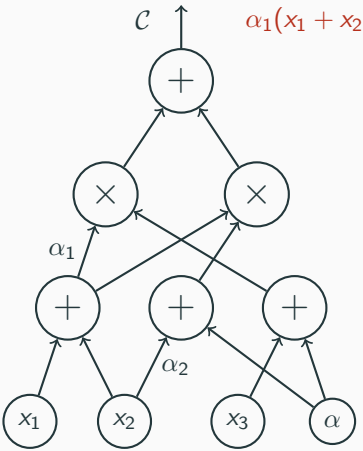

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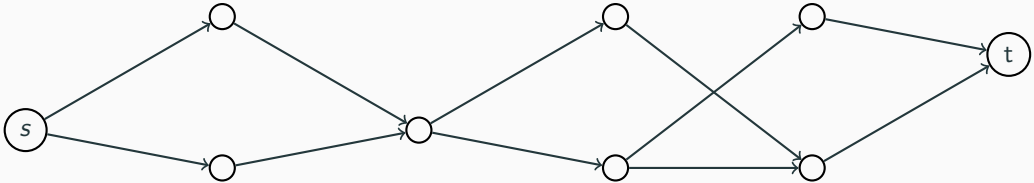
Central Question

$\boxed{\text{VP} \stackrel{?}{=} \text{VNP}}$: Find explicit polynomials that cannot be computed by circuits of size poly(n,d).

Algebraic Branching Programs



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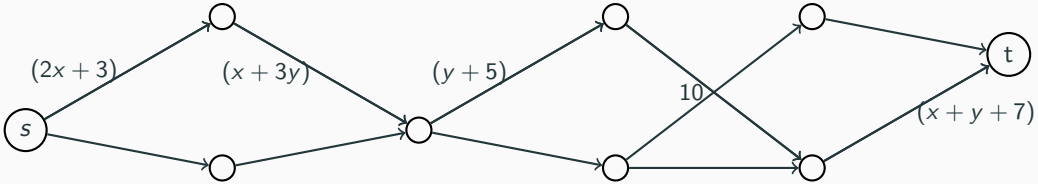

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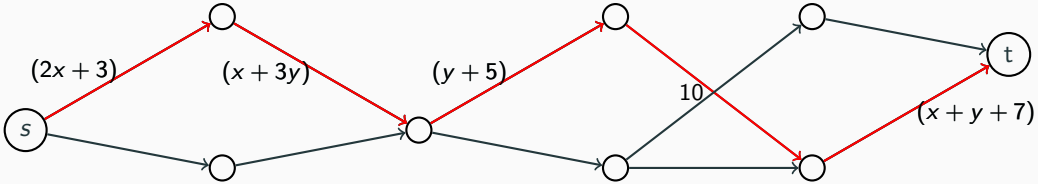

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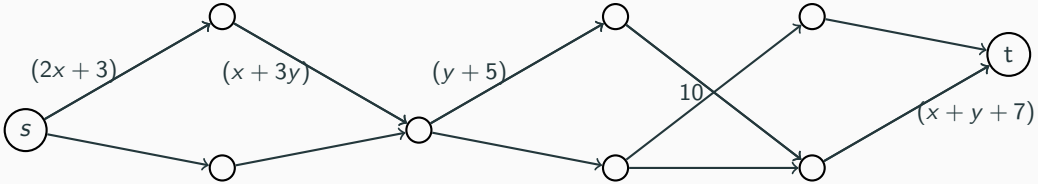

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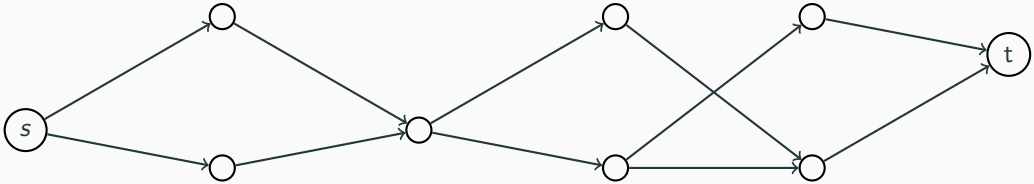

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In this talk: Is there an explicit n-variate, degree d polynomial that can not be represented by an ABP of size $\mathrm{poly}(n, d)$?

What is known?

For general ABPs, the best lower bound is just quadratic.

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[Bhargav-Dwivedi-Saxena]: Super polynomial lower bound against total-width of $\sum$osmABP for a polynomial of degree $O\left(\frac{\log n}{\log \log n}\right)$ implies super-polynomial lower bound against ABPs.

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The Question: Can we prove lower bounds against a general $\sum$osmABP?

Near Tightness of ABP Set-Multilinearisation

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Our Main Result: For $\omega(\log n) = d \leq n$, there is a polynomial $G_{n,d}(\mathbf{x})$ which is set-multilinear w.r.t $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$, where $|\mathbf{x}_i| \leq n$ for every $i \in [d]$, such that:

- $G_{n,d}$ is computable by a set-multilinear ABP of size poly(n),
- any $\sum$osmABP computing $G_{n,d}$ must have super-polynomial total-width.

Set-Multilinearity and Ordered Set-Multilinearity

Set-Multilinearity

The variable set is divided into buckets.

$$\mathbf{x} = \mathbf{x}_1 \cup \cdots \cup \mathbf{x}_d \quad \text{where} \quad \mathbf{x}_i = \{x_{i,1}, \ldots x_{i,n_i}\}.$$

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An ABP is set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if every path in it

computes a set-multilinear monomial with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$.

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Example:

$$\mathrm{IMM}_{n,d} = \sum_{1 \le i_1, \ldots, i_{d-1} \le n} x^{(1)}_{1,i_1} \cdot \left(\prod_{j=2}^{d-1} x^{(j)}_{i_{j-1}, i_j} \right) \cdot x^{(d)}_{i_{d-1}, i_d}.$$

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has an osmABP of size $O(nd)$ for $\sigma \in S_d$ being the identity permutation.

Our Main Result Revisited

An ABP is σ-ordered set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

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[C-Kush-Saraf-Shpilka]: Super polynomial lower bound against total-width of $\sum$osmABP for a polynomial of degree $d = \omega(\log n)$ that is computable by polynomial-sized ABPs.

Our Other Results

Exponential Lower Bound in the High Degree Regime

Sum of Ordered Set-Multilinear ABPs:
There is a polynomial $G_{n,n}(\mathbf{x})$ which is set-multilinear with respect to $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$, where $|\mathbf{x}_i| \leq n$ for each $i \in [n]$, such that:

- it has a set-multilinear branching program of size poly(n),
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A single Ordered Set-Multilinear ABP:
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- it has a set-multilinear branching program of size poly(n, d),
- but any ordered set-multilinear branching program computing $G_{n,d}$ requires width $n^{\Omega(d)}$.

Results for polynomials in VP, VNP

High Degree Regime: There is polynomial family $\{F_{n,n}(\mathbf{x})\}$, in VP, which is set-multilinear with respect to a set of $\Theta(n)$ buckets, each of size $\Theta(n)$, such that

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Related Results

[Bhargav-Dwivedi-Saxena]

Any $\sum$osmABP computing $\mathrm{IMM}_{n,n}$ which has max-width $n^{o(1)}$ must have $2^{\Omega(n)}$ summands.

Related Results

[Bhargav-Dwivedi-Saxena]

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[Arvind-Raja]

Any $\sum_{i=1}^{t}$ osmABP computing the $n \times n$ permanent polynomial has max-width $2^{\Omega(n/t)}$.

Related Results

[Ramya-Rao]

There exists an explicit polynomial family $\{g_n\}_n \in \mathsf{VP}$, g_n being defined on the variable set $\{x_{1,0}, x_{1,1}\} \cup \cdots \cup \{x_{n,0}, x_{n,1}\}$, such that any $\sum$osmABP computing it has total width $2^{\Omega\left(\frac{n^{1/6}}{\log n}\right)}$.

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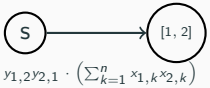
[Ghoshal-Rao]

There exists an explicit polynomial family $\{g_n\}_n \in \mathrm{VBP}$, g_n being defined on the variable set $\{x_{1,0}, x_{1,1}\} \cup \cdots \cup \{x_{n,0}, x_{n,1}\}$, such that any $\sum$osmABP computing g_n that has max-width poly(n) must have total width $2^{\Omega\left(n^{1/500}\right)}$.

Proof Overviews

The Hard Polynomial



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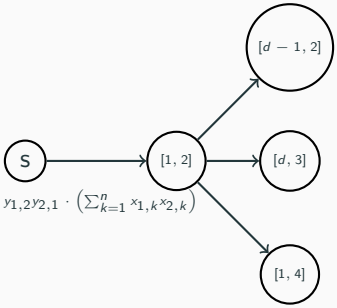

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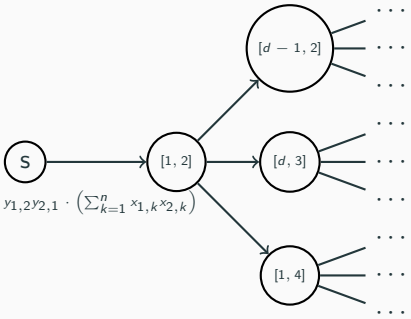

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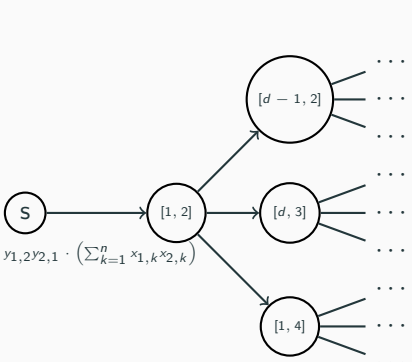

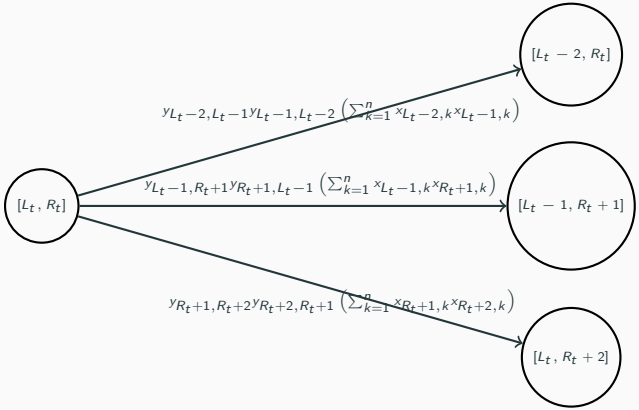

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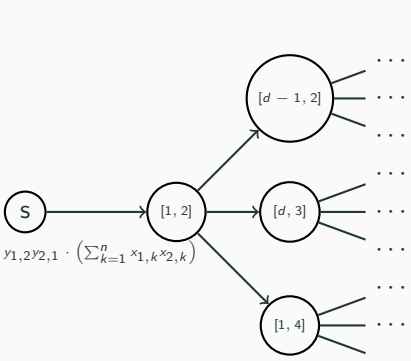
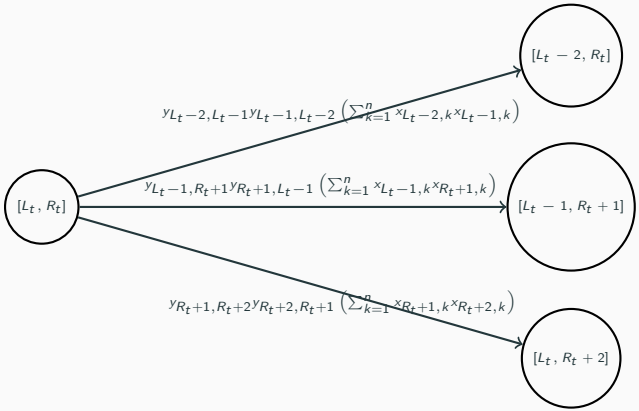
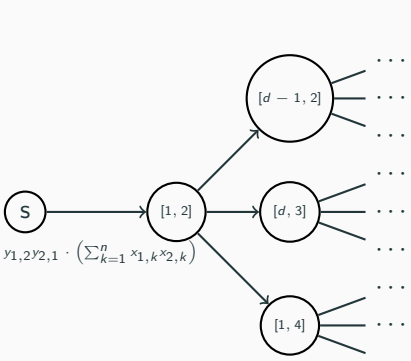
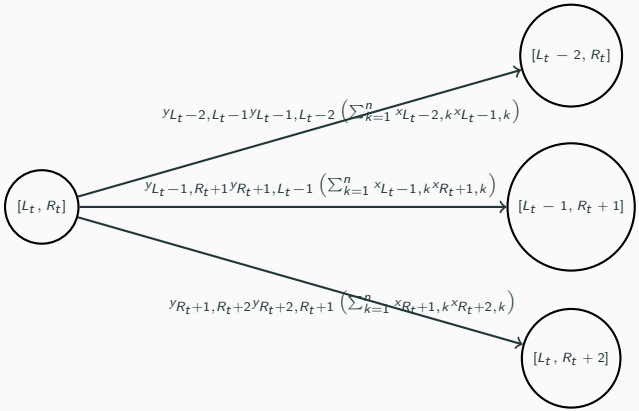




Every path corresponds to a sequence of $d/2$ pairs.

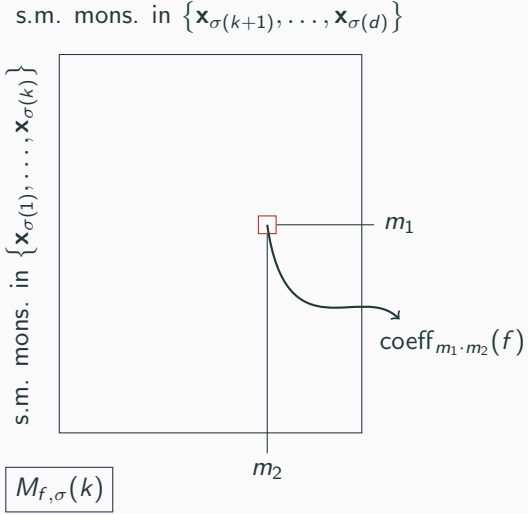
The Hard Polynomial





Every path corresponds to a sequence of $d/2$ pairs. $\mathcal{P}_{d/2}$: Set of all such sequences of pairs.

Lower Bound for a single osmABP



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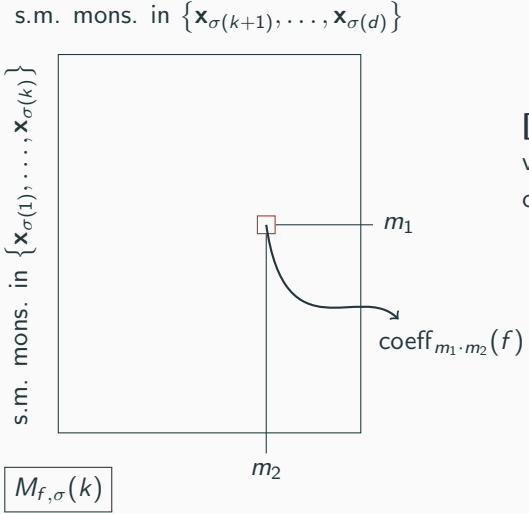

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[Nisan]: For every $1 \leq k \leq d$, the number of vertices in the k-th layer of the smallest osmABP(σ) computing f is equal to the rank of $M_{f,\sigma}(k)$.

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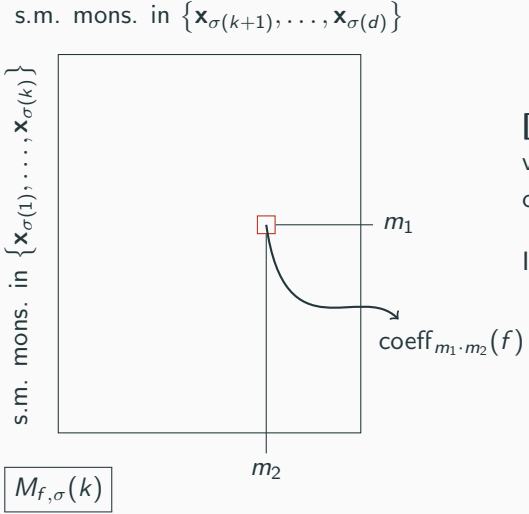

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If $\mathcal{A}$ is the smallest osmABP computing f, then

$$\text{size}(\mathcal{A}) = \sum_{i=1}^{d} \text{rank}(M_{f,\sigma}(k)).$$

Lower Bound for a single osmABP (contd.)

$$G_{n,d} = \sum_{\mathcal{P} \in \mathcal{P}_{d/2}} \prod_{(i,j) \in \mathcal{P}} y_{i,j} y_{j,i} \cdot \left(\sum_{k=1}^{n} x_{i,k} x_{j,k} \right).$$

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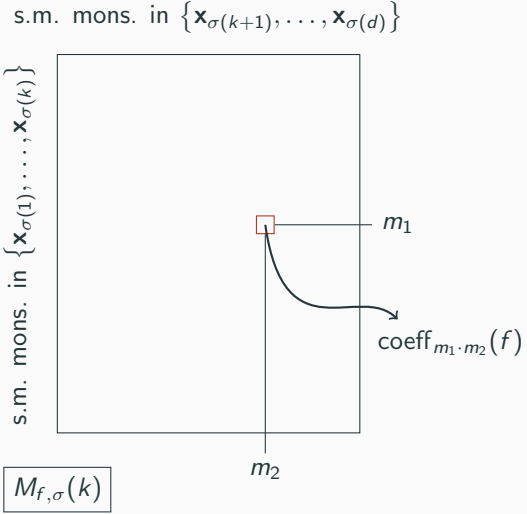

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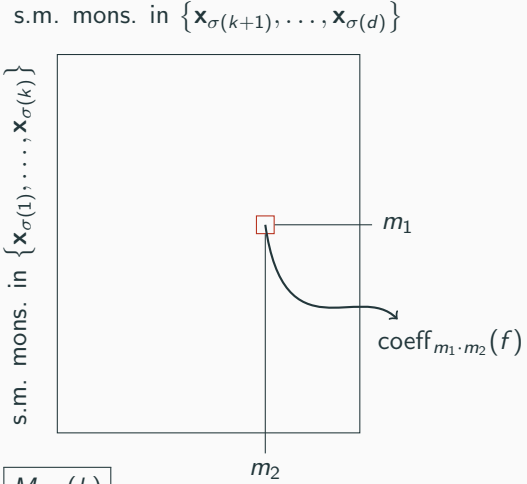

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  Therefore,

  $$\text{rank}(M_{G_{n,d},\sigma}(d/2)) = \Omega(n^{d/8}).$$

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Open Questions

Open Threads

1. PIT for $\sum$osmABP?

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Thank you!