Lower Bounds for some Algebraic Models of Computation

Prerona Chatterjee

April 10, 2024

Complexity Theory

Q: Given a computational problem and constraints on the computational power at hand,

Complexity Theory

Q: Given a computational problem and constraints on the computational power at hand,

- design a computational model that captures the constraints

Complexity Theory

Q: Given a computational problem and constraints on the computational power at hand,

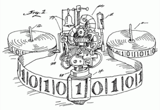
- design a computational model that captures the constraints
- study the amount of resource required by the model to complete the task.

Q: Given a computational problem and constraints on the computational power at hand,

- design a computational model that captures the constraints
- study the amount of resource required by the model to complete the task.

Traditional Time Complexity

Given a boolean function f on n inputs, how many steps are required by a Turing machine to compute the f (in terms of n)?



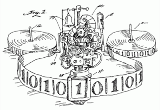
Complexity Theory

Q: Given a computational problem and constraints on the computational power at hand,

- design a computational model that captures the constraints
- study the amount of resource required by the model to complete the task.

Traditional Time Complexity

Given a boolean function f on n inputs, how many steps are required by a Turing machine to compute the f (in terms of n)?



Traditional Space Complexity

Given a boolean function f on n inputs, how much space is required by a Turing machine to compute the f (in terms of n)?

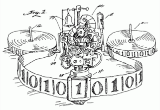
Complexity Theory

Q: Given a computational problem and constraints on the computational power at hand,

- design a computational model that captures the constraints
- study the amount of resource required by the model to complete the task.

Traditional Time Complexity

Given a boolean function f on n inputs, how many steps are required by a Turing machine to compute the f (in terms of n)?



Traditional Space Complexity

Given a boolean function f on n inputs, how much space is required by a Turing machine to compute the f (in terms of n)?

Circuit Complexity

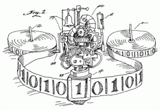
Complexity Theory

Q: Given a computational problem and constraints on the computational power at hand,

- design a computational model that captures the constraints
- study the amount of resource required by the model to complete the task.

Traditional Time Complexity

Given a boolean function f on n inputs, how many steps are required by a Turing machine to compute the f (in terms of n)?



Traditional Space Complexity

Given a boolean function f on n inputs, how much space is required by a Turing machine to compute the f (in terms of n)?

Circuit Complexity

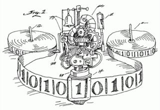
Communication Complexity

Complexity Theory

Q: Given a computational problem and constraints on the computational power at hand,

- design a computational model that captures the constraints
- study the amount of resource required by the model to complete the task.

Traditional Time Complexity
Given a boolean function f on n inputs, how many steps are required by a Turing machine to compute the f (in terms of n)?



Traditional Space Complexity
Given a boolean function f on n inputs, how much space is required by a Turing machine to compute the f (in terms of n)?

Circuit Complexity | **Communication Complexity** | **Quantum Complexity**

Complexity of Computing Polynomials

Q: Given $f(\mathbf{x}) \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d, how many additions and multiplications does it take to compute f formally?

Complexity of Computing Polynomials

Q: Given $f(\mathbf{x}) \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d, how many additions and multiplications does it take to compute f formally?

Why? More tools to work with.

Complexity of Computing Polynomials

Q: Given $f(\mathbf{x}) \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d, how many additions and multiplications does it take to compute f formally?

Why? More tools to work with.

Usually, Upper Bounds in this setting $\implies$ Upper Bounds in the boolean setting.

Complexity of Computing Polynomials

Q: Given $f(\mathbf{x}) \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d, how many additions and multiplications does it take to compute f formally?

Why? More tools to work with.

Usually, Upper Bounds in this setting $\implies$ Upper Bounds in the boolean setting.

Lower Bound in this setting is like a step towards Lower Bound in the boolean setting.

Complexity of Computing Polynomials

Q: Given $f(\mathbf{x}) \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d, how many additions and multiplications does it take to compute f formally?

Why? More tools to work with.

Usually, Upper Bounds in this setting $\implies$ Upper Bounds in the boolean setting.

Lower Bound in this setting is like a step towards Lower Bound in the boolean setting.

[Shamir 79, Lipton 94]: If $h(x) = \prod_{i=1}^{d}(x - i)$ can be computed using $\text{poly}(\log d)$ additions and multiplications, then integer factoring is easy for boolean circuits.

Complexity of Computing Polynomials

Q: Given $f(\mathbf{x}) \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d, how many additions and multiplications does it take to compute f formally?

Why? More tools to work with.

Usually, Upper Bounds in this setting $\implies$ Upper Bounds in the boolean setting.

Lower Bound in this setting is like a step towards Lower Bound in the boolean setting.

[Shamir 79, Lipton 94]: If $h(x) = \prod_{i=1}^{d}(x - i)$ can be computed using $\mathrm{poly}(\log d)$ additions and multiplications, then integer factoring is easy for boolean circuits.

Why? Polynomials are central to many algoritms.

Complexity of Computing Polynomials

Q: Given $f(\mathbf{x}) \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d, how many additions and multiplications does it take to compute f formally?

Why? More tools to work with.

Usually, Upper Bounds in this setting $\implies$ Upper Bounds in the boolean setting.

Lower Bound in this setting is like a step towards Lower Bound in the boolean setting.

[Shamir 79, Lipton 94]: If $h(x) = \prod_{i=1}^{d}(x - i)$ can be computed using $\mathrm{poly}(\log d)$ additions and multiplications, then integer factoring is easy for boolean circuits.

Why? Polynomials are central to many algoritms.

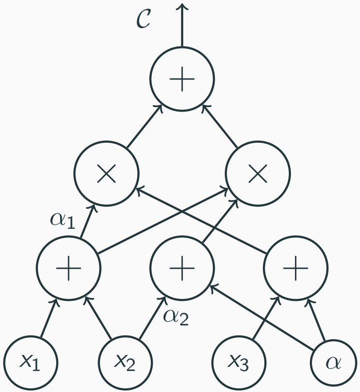
Matrix Multiplication Exponent (ω): Smallest number k such that the product of two $n \times n$ matrices can be found using n^k multiplications.

Algebraic Models of Computation

Q: Given $f(\mathbf{x}) \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d, how many $+, \times, -$ gates are needed to compute f?

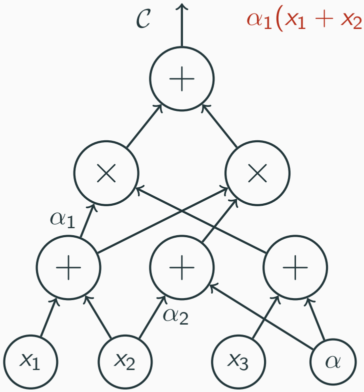
Algebraic Models of Computation

Q: Given $f(\mathbf{x}) \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d, how many $+, \times, -$ gates are needed to compute f?



Algebraic Models of Computation

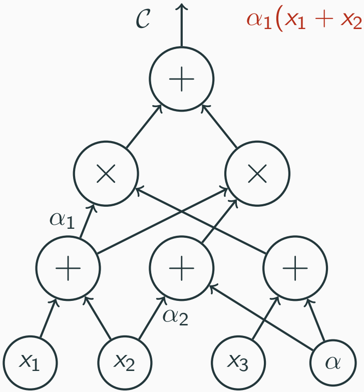
Q: Given $f(\mathbf{x}) \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d, how many $+, \times, -$ gates are needed to compute f?



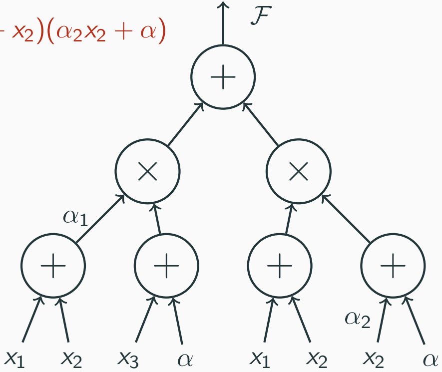
$\alpha_1(x_1 + x_2)(x_3 + \alpha) + (x_1 + x_2)(\alpha_2 x_2 + \alpha)$

Algebraic Models of Computation

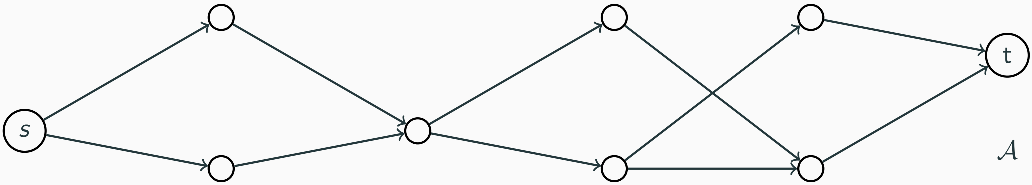
Q: Given $f(\mathbf{x}) \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d, how many $+, \times, -$ gates are needed to compute f?



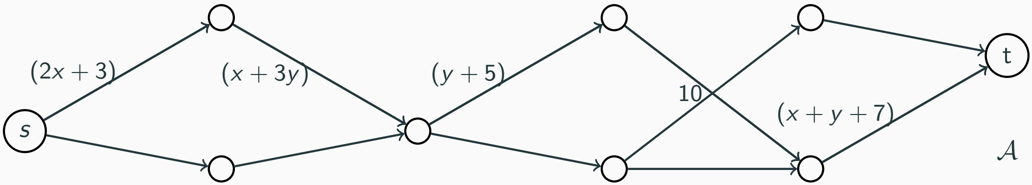
$\alpha_1(x_1 + x_2)(x_3 + \alpha) + (x_1 + x_2)(\alpha_2 x_2 + \alpha)$



Algebraic Branching Programs

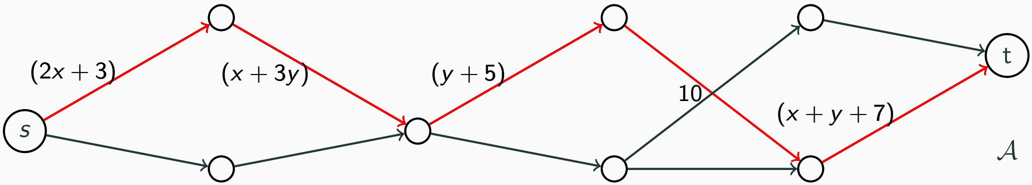


Algebraic Branching Programs



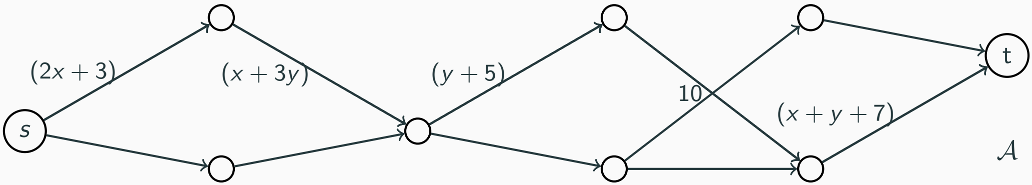
- Label on each edge: An affine linear form in $\{x_1, x_2, \ldots, x_n\}$

Algebraic Branching Programs



- Label on each edge: An affine linear form in $\{x_1, x_2, \ldots, x_n\}$
- Polynomial computed by the path $p = \mathrm{wt}(p)$: Product of the edge labels on p

Algebraic Branching Programs



- Label on each edge: An affine linear form in $\{x_1, x_2, \ldots, x_n\}$
- Polynomial computed by the path $p = \mathrm{wt}(p)$: Product of the edge labels on p
- Polynomial computed by the ABP: $f_{\mathcal{A}}(\mathbf{x}) = \sum_p \mathrm{wt}(p)$

Objects of Study: Polynomials over n variables of degree d.

Lower Bounds in Algebraic Circuit Complexity

Objects of Study: Polynomials over n variables of degree d.

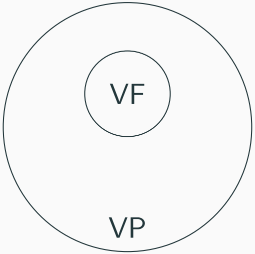
VP: Polynomials computable by circuits of size $\text{poly}(n, d)$.

Lower Bounds in Algebraic Circuit Complexity

Objects of Study: Polynomials over n variables of degree d.

VF: Polynomials computable by formulas of size $\mathrm{poly}(n, d)$.

VP: Polynomials computable by circuits of size $\mathrm{poly}(n, d)$.



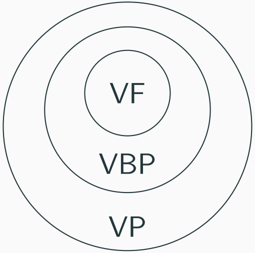
Lower Bounds in Algebraic Circuit Complexity

Objects of Study: Polynomials over n variables of degree d.

VF: Polynomials computable by formulas of size poly(n, d).

VBP: Polynomials computable by ABPs of size poly(n, d).

VP: Polynomials computable by circuits of size poly(n, d).



Lower Bounds in Algebraic Circuit Complexity

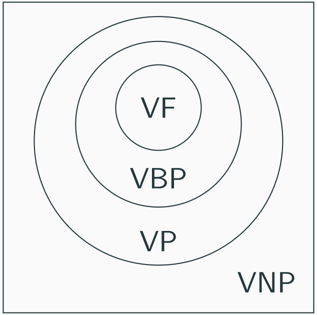
Objects of Study: Polynomials over n variables of degree d.

VF: Polynomials computable by formulas of size poly(n, d).

VBP: Polynomials computable by ABPs of size poly(n, d).

VP: Polynomials computable by circuits of size poly(n, d).

VNP: Explicit Polynomials



Lower Bounds in Algebraic Circuit Complexity

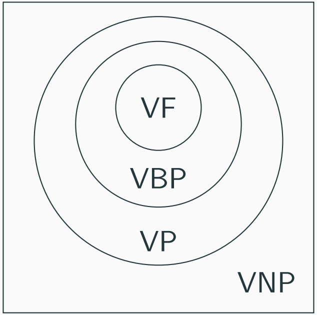
Objects of Study: Polynomials over n variables of degree d.

VF: Polynomials computable by formulas of size $\text{poly}(n, d)$.

VBP: Polynomials computable by ABPs of size $\text{poly}(n, d)$.

VP: Polynomials computable by circuits of size $\text{poly}(n, d)$.

VNP: Explicit Polynomials



Central Question: Find explicit polynomials that cannot be computed by efficient circuits.

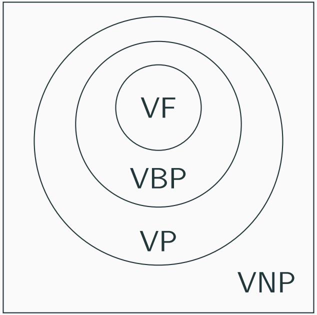
Objects of Study: Polynomials over n variables of degree d.

VF: Polynomials computable by formulas of size $\mathrm{poly}(n, d)$.

VBP: Polynomials computable by ABPs of size $\mathrm{poly}(n, d)$.

VP: Polynomials computable by circuits of size $\mathrm{poly}(n, d)$.

VNP: Explicit Polynomials



$$\boxed{\mathrm{VP} = \mathrm{VNP} \overset{\text{G.R.H.}}{\Longrightarrow} \mathrm{P} = \mathrm{NP}}$$

Central Question: Find explicit polynomials that cannot be computed by efficient circuits.

Lower Bounds in Algebraic Circuit Complexity

Objects of Study: Polynomials over n variables of degree d.

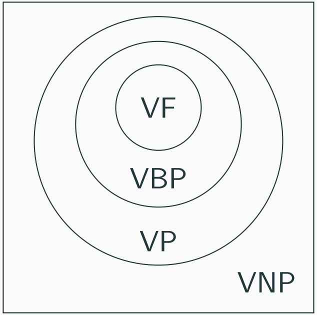
VF: Polynomials computable by formulas of size $\text{poly}(n, d)$.

VBP: Polynomials computable by ABPs of size $\text{poly}(n, d)$.

VP: Polynomials computable by circuits of size $\text{poly}(n, d)$.

VNP: Explicit Polynomials

$$\boxed{\text{VP} = \text{VNP} \overset{\text{G.R.H.}}{\Longrightarrow} \text{P} = \text{NP}}$$



Central Question: Find explicit polynomials that cannot be computed by efficient circuits.

Other Motivating Questions: Are the other inclusions tight?

Lower Bounds for General Models

General Circuits

[Baur-Strassen 83]: Any algebraic circuit computing $\sum_{i=1}^{n} x_i^d$ requires $\Omega(n \log d)$ wires.

Lower Bounds for General Models

General Circuits

[Baur-Strassen 83]: Any algebraic circuit computing $\sum_{i=1}^{n} x_i^d$ requires $\Omega(n \log d)$ wires.

General ABPs

[C-Kumar-She-Volk 22]: Any ABP computing $\sum_{i=1}^{n} x_i^d$ requires $\Omega(nd)$ vertices.

Lower Bounds for General Models

General Circuits

[Baur-Strassen 83]: Any algebraic circuit computing $\sum_{i=1}^{n} x_i^d$ requires $\Omega(n \log d)$ wires.

General ABPs

[C-Kumar-She-Volk 22]: Any ABP computing $\sum_{i=1}^{n} x_i^d$ requires $\Omega(nd)$ vertices.

General Formulas

[Kalorkoti 85]: Any formula computing the n^2-variate $\mathrm{Det}_n(\mathbf{x})$ requires $\Omega(n^3)$ wires.

Lower Bounds for General Models

General Circuits

[Baur-Strassen 83]: Any algebraic circuit computing $\sum_{i=1}^{n} x_i^d$ requires $\Omega(n \log d)$ wires.

General ABPs

[C-Kumar-She-Volk 22]: Any ABP computing $\sum_{i=1}^{n} x_i^d$ requires $\Omega(nd)$ vertices.

General Formulas

[Kalorkoti 85]: Any formula computing the n^2-variate $\mathrm{Det}_n(\mathbf{x})$ requires $\Omega(n^3)$ wires.

[Shpilka-Yehudayoff 10] (using Kalorkoti's method): There is an n-variate multilinear polynomial such that any formula computing it requires $\Omega(n^2/\log n)$ wires.

Lower Bounds for General Models

General Circuits

[Baur-Strassen 83]: Any algebraic circuit computing $\sum_{i=1}^{n} x_i^d$ requires $\Omega(n \log d)$ wires.

General ABPs

[C-Kumar-She-Volk 22]: Any ABP computing $\sum_{i=1}^{n} x_i^d$ requires $\Omega(nd)$ vertices.

General Formulas

[Kalorkoti 85]: Any formula computing the n^2-variate $\mathrm{Det}_n(\mathbf{x})$ requires $\Omega(n^3)$ wires.

[Shpilka-Yehudayoff 10] (using Kalorkoti's method): There is an n-variate multilinear polynomial such that any formula computing it requires $\Omega(n^2/\log n)$ wires.

[C-Kumar-She-Volk 22]: Any formula computing $\mathrm{ESYM}_{n,0.1n}(\mathbf{x})$ requires $\Omega(n^2)$ vertices.

$$\mathrm{ESYM}_{n,d}(\mathbf{x}) = \sum_{i_1 < \cdots < i_d \in [n]} x_{i_1} \cdots x_{i_d}.$$

How does one make progress?

Structural Results

Show that if a structured n-variate, degree-d polynomial is computable by a general model of size s, then they can also be computed by a structured model of size $\mathrm{func}(s, n, d)$ for some function func.

How does one make progress?

Structural Results

Show that if a structured n-variate, degree-d polynomial is computable by a general model of size s, then they can also be computed by a structured model of size $\text{func}(s, n, d)$ for some function func.

Study Structured Models

Prove strong lower bounds against structured models computing f.

How does one make progress?

Structural Results

Show that if a structured n-variate, degree-d polynomial is computable by a general model of size s, then they can also be computed by a structured model of size $\text{func}(s, n, d)$ for some function func.

[Agrawal-Vinay 08, Koiran 12, Tavenas 15]

Size s circuits computing n-variate degree d polynomials can be converted into depth-4 circuits of size $s^{O(\sqrt{d})}$.

Study Structured Models

Prove strong lower bounds against structured models computing f.

How does one make progress?

Structural Results

Show that if a structured n-variate, degree-d polynomial is computable by a general model of size s, then they can also be computed by a structured model of size $\text{func}(s, n, d)$ for some function func.

Study Structured Models

Prove strong lower bounds against structured models computing f.

[Agrawal-Vinay 08, Koiran 12, Tavenas 15]

Size s circuits computing n-variate degree d polynomials can be converted into depth-4 circuits of size $s^{O(\sqrt{d})}$.

[Gupta-Kamath-Kayal-Saptharishi 16]

Size s circuits computing n-variate degree d polynomials can be converted into depth-3 circuits of size $s^{O(\sqrt{d})}$.

How does one make progress?

Structural Results

Show that if a structured n-variate, degree-d polynomial is computable by a general model of size s, then they can also be computed by a structured model of size $\text{func}(s, n, d)$ for some function func.

Study Structured Models

Prove strong lower bounds against structured models computing f.

[Agrawal-Vinay 08, Koiran 12, Tavenas 15]

Size s circuits computing n-variate degree d polynomials can be converted into depth-4 circuits of size $s^{O(\sqrt{d})}$.

[Gupta-Kamath-Kayal-Saptharishi 16]

Size s circuits computing n-variate degree d polynomials can be converted into depth-3 circuits of size $s^{O(\sqrt{d})}$.

A lot of work that culminated in

[Limaye-Srinivasan-Tavenas 24]

Any constant depth circuit computing $\mathrm{IMM}_{n,\log n}(\mathbf{x})$ must have super-polynomial size.

How does one make progress?

Structural Results

Show that if a structured n-variate, degree-d polynomial is computable by a general model of size s, then they can also be computed by a structured model of size $\text{func}(s, n, d)$ for some function func.

[Agrawal-Vinay 08, Koiran 12, Tavenas 15]

Size s circuits computing n-variate degree d polynomials can be converted into depth-4 circuits of size $s^{O(\sqrt{d})}$.

[Gupta-Kamath-Kayal-Saptharishi 16]

Size s circuits computing n-variate degree d polynomials can be converted into depth-3 circuits of size $s^{O(\sqrt{d})}$.

Study Structured Models

Prove strong lower bounds against structured models computing f.

A lot of work that culminated in

[Limaye-Srinivasan-Tavenas 24]

Any constant depth circuit computing $\mathrm{IMM}_{n,\log n}(\mathbf{x})$ must have super-polynomial size.
The lower bound is $n^{\Omega(\sqrt{d})}$ for depth-3 and depth-4.

Towards Better ABP Lower Bounds

[C-Kumar-She-Volk 22]: Any ABP computing $\sum_{i=1}^{n} x_i^d$ requires $\Omega(nd)$ vertices.

Towards Better ABP Lower Bounds

[C-Kumar-She-Volk 22]: Any ABP computing $\sum_{i=1}^{n} x_i^d$ requires $\Omega(nd)$ vertices.

[Bhargav-Dwivedi-Saxena 24]: Super polynomial lower bound against total-width of $\sum$ osmABP for a polynomial of degree $d = O\left(\frac{\log n}{\log \log n}\right) \implies$ super-polynomial lower bound against ABPs.

Towards Better ABP Lower Bounds

[C-Kumar-She-Volk 22]: Any ABP computing $\sum_{i=1}^{n} x_i^d$ requires $\Omega(nd)$ vertices.

[Bhargav-Dwivedi-Saxena 24]: Super polynomial lower bound against total-width of $\sum$ osmABP for a polynomial of degree $d = O\left(\frac{\log n}{\log \log n}\right)$ $\implies$ super-polynomial lower bound against ABPs.

[C-Kush-Saraf-Shpilka 24]: For $\omega(\log n) = d \leq n$, there is a polynomial $G_{n,d}(\mathbf{x})$ which is set-multilinear w.r.t $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$, where $|\mathbf{x}_i| \leq n$ for every $i \in [d]$, such that:

- $G_{n,d}$ is computable by a set-multilinear ABP of size $\text{poly}(n)$,
- any $\sum$ osmABP computing $G_{n,d}$ must have super-polynomial total-width.

Set-Multilinearity

The variable set is divided into buckets.

$$\mathbf{x} = \mathbf{x}_1 \cup \cdots \cup \mathbf{x}_d \quad \text{where} \quad \mathbf{x}_i = \left\{x_{i,1}, \ldots x_{i,n_i}\right\}.$$

Set-Multilinearity

The variable set is divided into buckets.

$$\mathbf{x} = \mathbf{x}_1 \cup \cdots \cup \mathbf{x}_d \quad \text{where} \quad \mathbf{x}_i = \{x_{i,1}, \ldots x_{i,n_i}\}.$$

f is set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

every monomial in f has exactly one variable from $\mathbf{x}_i$ for each $i \in [d]$.

Set-Multilinearity

The variable set is divided into buckets.

$$\mathbf{x} = \mathbf{x}_1 \cup \cdots \cup \mathbf{x}_d \quad \text{where} \quad \mathbf{x}_i = \{x_{i,1}, \ldots x_{i,n_i}\}.$$

f is set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

every monomial in f has exactly one variable from $\mathbf{x}_i$ for each $i \in [d]$.

An ABP is set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if every path in it

computes a set-multilinear monomial with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$.

Near Tightness of ABP Set-Multilinearisation

For $\sigma \in S_d$, an ABP is σ-ordered set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

- there are d layers in the ABP
- every edge in layer i is labelled by a homogeneous linear form in $\mathbf{x}_{\sigma(i)}$

Near Tightness of ABP Set-Multilinearisation

For $\sigma \in S_d$, an ABP is σ-ordered set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

- there are d layers in the ABP
- every edge in layer i is labelled by a homogeneous linear form in $\mathbf{x}_{\sigma(i)}$

$\sum$osmABP: Sum of ordered set-multilinear ABPs, each with a possibly different ordering.

Near Tightness of ABP Set-Multilinearisation

For $\sigma \in S_d$, an ABP is σ-ordered set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

- there are d layers in the ABP
- every edge in layer i is labelled by a homogeneous linear form in $\mathbf{x}_{\sigma(i)}$

$\sum$osmABP: Sum of ordered set-multilinear ABPs, each with a possibly different ordering.

[B-D-S 24]: Super polynomial lower bound against total-width of $\sum$osmABP for a polynomial of degree $d = O\left(\frac{\log n}{\log \log n}\right) \implies$ super-polynomial lower bound against ABPs.

Near Tightness of ABP Set-Multilinearisation

For $\sigma \in S_d$, an ABP is σ-ordered set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

- there are d layers in the ABP
- every edge in layer i is labelled by a homogeneous linear form in $\mathbf{x}_{\sigma(i)}$

$\sum$osmABP: Sum of ordered set-multilinear ABPs, each with a possibly different ordering.

[B-D-S 24]: Super polynomial lower bound against total-width of $\sum$osmABP for a polynomial of degree $d = O\left(\frac{\log n}{\log \log n}\right) \implies$ super-polynomial lower bound against ABPs.

[C-K-S-S 24]: Super polynomial lower bound against total-width of $\sum$osmABP for a polynomial of degree $d = \omega(\log n)$ that is computable by polynomial-sized ABPs.

Non-Commutativity

$$f(x, y) = (x + y) \times (x + y) = x^2 + xy + yx + y^2 \neq x^2 + 2xy + y^2$$

Non-Commutativity

$$f(x, y) = (x + y) \times (x + y) = x^2 + xy + yx + y^2 \neq x^2 + 2xy + y^2$$

Non-Commutative Models: The multiplication gates, additionally, respect the order.

Non-Commutativity

$$f(x, y) = (x + y) \times (x + y) = x^2 + xy + yx + y^2 \neq x^2 + 2xy + y^2$$

Non-Commutative Models: The multiplication gates, additionally, respect the order.

Can we do better in this setting?

Non-Commutativity

$$f(x, y) = (x + y) \times (x + y) = x^2 + xy + yx + y^2 \neq x^2 + 2xy + y^2$$

Non-Commutative Models: The multiplication gates, additionally, respect the order.

Can we do better in this setting? For general circuits, continues to be $\Omega(n \log d)$.

Non-Commutativity

$$f(x,y) = (x+y) \times (x+y) = x^2 + xy + yx + y^2 \neq x^2 + 2xy + y^2$$

Non-Commutative Models: The multiplication gates, additionally, respect the order.

Can we do better in this setting? For general circuits, continues to be $\Omega(n \log d)$.

[C-Hrubeš 23]: Any homogeneous non-commutative circuit computing

$$\mathrm{OSym}_{n,d}(\mathbf{x}) = \sum_{1 \leq i_1 < \cdots < i_d \leq n} x_{i_1} \cdots x_{i_d}$$

has size $\Omega(nd)$ for $d \leq \frac{n}{2}$.

Non-Commutativity

$$f(x,y) = (x+y) \times (x+y) = x^2 + xy + yx + y^2 \neq x^2 + 2xy + y^2$$

Non-Commutative Models: The multiplication gates, additionally, respect the order.

Can we do better in this setting? For general circuits, continues to be $\Omega(n \log d)$.

[C-Hrubeš 23]: Any homogeneous non-commutative circuit computing

$$\mathrm{OSym}_{n,d}(\mathbf{x}) = \sum_{1 \leq i_1 < \cdots < i_d \leq n} x_{i_1} \cdots x_{i_d}$$

has size $\Omega(nd)$ for $d \leq \frac{n}{2}$. The lower bound is tight for homogeneous non-commutative circuits.

Non-Commutativity

$$f(x, y) = (x + y) \times (x + y) = x^2 + xy + yx + y^2 \neq x^2 + 2xy + y^2$$

Non-Commutative Models: The multiplication gates, additionally, respect the order.

Can we do better in this setting? For general circuits, continues to be $\Omega(n \log d)$.

[C-Hrubeš 23]: Any homogeneous non-commutative circuit computing

$$\mathrm{OSym}_{n,d}(\mathbf{x}) = \sum_{1 \leq i_1 < \cdots < i_d \leq n} x_{i_1} \cdots x_{i_d}$$

has size $\Omega(nd)$ for $d \leq \frac{n}{2}$. The lower bound is tight for homogeneous non-commutative circuits.

Further, there is a non-commutative circuit of size $O(n \log^2 n)$ that computes $\mathrm{OSym}_{n,n/2}(\mathbf{x})$.

ABP vs Formula in the Non-Commutative Setting

[Nisan 91]: Any ABP computing $\mathrm{Pal}_n(x_0, x_1) = \sum_{w \in \{0,1\}^{n/2}} \mathbf{x}_w \cdot \mathbf{x}_{w^R}$ has size $2^{\Omega(n)}$.

ABP vs Formula in the Non-Commutative Setting

[Nisan 91]: Any ABP computing $\mathrm{Pal}_n(x_0, x_1) = \sum_{w \in \{0,1\}^{n/2}} \mathbf{x}_w \cdot \mathbf{x}_{w^R}$ has size $2^{\Omega(n)}$.

[Tavenas-Limaye-Srinivasan 22]: Any homogeneous non-commutative formula computing $\mathrm{IMM}_{n,d}(\mathbf{x})$ has size $n^{\Omega(\log \log d)}$.

ABP vs Formula in the Non-Commutative Setting

[Nisan 91]: Any ABP computing $\mathrm{Pal}_n(x_0, x_1) = \sum_{w \in \{0,1\}^{n/2}} \mathbf{x}_w \cdot \mathbf{x}_{w^R}$ has size $2^{\Omega(n)}$.

[Tavenas-Limaye-Srinivasan 22]: Any homogeneous non-commutative formula computing $\mathrm{IMM}_{n,d}(\mathbf{x})$ has size $n^{\Omega(\log \log d)}$.

homogeneous non-commutative ABPs, formulas $\equiv$ ordered set-multilinear ABPs, formulas

ABP vs Formula in the Non-Commutative Setting

[Nisan 91]: Any ABP computing $\mathrm{Pal}_n(x_0, x_1) = \sum_{w \in \{0,1\}^{n/2}} \mathbf{x}_w \cdot \mathbf{x}_{w^R}$ has size $2^{\Omega(n)}$.

[Tavenas-Limaye-Srinivasan 22]: Any homogeneous non-commutative formula computing $\mathrm{IMM}_{n,d}(\mathbf{x})$ has size $n^{\Omega(\log \log d)}$.

homogeneous non-commutative ABPs, formulas $\equiv$ ordered set-multilinear ABPs, formulas

$$x_1 x_2 + x_2 x_1 \longrightarrow x_{1,1} x_{2,2} + x_{1,2} x_{2,1}$$

ABP vs Formula in the Non-Commutative Setting

[Nisan 91]: Any ABP computing $\mathrm{Pal}_n(x_0, x_1) = \sum_{w \in \{0,1\}^{n/2}} \mathbf{x}_w \cdot \mathbf{x}_{w^R}$ has size $2^{\Omega(n)}$.

[Tavenas-Limaye-Srinivasan 22]: Any homogeneous non-commutative formula computing $\mathrm{IMM}_{n,d}(\mathbf{x})$ has size $n^{\Omega(\log\log d)}$.

homogeneous non-commutative ABPs, formulas $\equiv$ ordered set-multilinear ABPs, formulas

$$x_1 x_2 + x_2 x_1 \longrightarrow x_{1,1} x_{2,2} + x_{1,2} x_{2,1}$$

$$x_2 x_3 + x_1 x_2 \longleftarrow x_{1,2} x_{2,3} + x_{1,1} x_{2,2}$$

ABP vs Formula in the Non-Commutative Setting

[Nisan 91]: Any ABP computing $\mathrm{Pal}_n(x_0, x_1) = \sum_{w \in \{0,1\}^{n/2}} \mathbf{x}_w \cdot \mathbf{x}_{w^R}$ has size $2^{\Omega(n)}$.

[Tavenas-Limaye-Srinivasan 22]: Any homogeneous non-commutative formula computing $\mathrm{IMM}_{n,d}(\mathbf{x})$ has size $n^{\Omega(\log \log d)}$.

homogeneous non-commutative ABPs, formulas $\equiv$ ordered set-multilinear ABPs, formulas

$$x_1 x_2 + x_2 x_1 \longrightarrow x_{1,1} x_{2,2} + x_{1,2} x_{2,1}$$

$$x_2 x_3 + x_1 x_2 \longleftarrow x_{1,2} x_{2,3} + x_{1,1} x_{2,2}$$

position indices $\equiv$ bucket indices

Tight Separation in a Structured Setting

$\{X_1, \ldots, X_m\}$: Partition of the underlying set of variables $\{x_1, \ldots, x_n\}$.

Tight Separation in a Structured Setting

$\{X_1, \ldots, X_m\}$: Partition of the underlying set of variables $\{x_1, \ldots, x_n\}$.

Ordered Set-Multilinear Polynomials: Every monomial has the form $X_1 X_2 \cdots X_m$.

Tight Separation in a Structured Setting

$\{X_1, \ldots, X_m\}$: Partition of the underlying set of variables $\{x_1, \ldots, x_n\}$.

Ordered Set-Multilinear Polynomials: Every monomial has the form $X_1 X_2 \cdots X_m$.

Abecedarian Polynomials: Every monomial has the form $X_1^* X_2^* \cdots X_m^*$.

Tight Separation in a Structured Setting

$\{X_1, \ldots, X_m\}$: Partition of the underlying set of variables $\{x_1, \ldots, x_n\}$.

Ordered Set-Multilinear Polynomials: Every monomial has the form $X_1 X_2 \cdots X_m$.

Abecedarian Polynomials: Every monomial has the form $X_1^* X_2^* \cdots X_m^*$.

Abecedarian Formulas: Every gate can be labelled by bucket indices of the end points.

Tight Separation in a Structured Setting

$\{X_1, \ldots, X_m\}$: Partition of the underlying set of variables $\{x_1, \ldots, x_n\}$.

Ordered Set-Multilinear Polynomials: Every monomial has the form $X_1 X_2 \cdots X_m$.

Abecedarian Polynomials: Every monomial has the form $X_1^* X_2^* \cdots X_m^*$.

Abecedarian Formulas: Every gate can be labelled by bucket indices of the end points.

[**Cha 21**]: For $\mathbf{x} = \cup_{i \in [n]} \{X_i\}$ with $X_i = \{x_{i,j}\}_{j \in [n]}$,

Tight Separation in a Structured Setting

$\{X_1, \ldots, X_m\}$: Partition of the underlying set of variables $\{x_1, \ldots, x_n\}$.

Ordered Set-Multilinear Polynomials: Every monomial has the form $X_1 X_2 \cdots X_m$.

Abecedarian Polynomials: Every monomial has the form $X_1^* X_2^* \cdots X_m^*$.

Abecedarian Formulas: Every gate can be labelled by bucket indices of the end points.

[**Cha 21**]: For $\mathbf{x} = \cup_{i \in [n]} \{X_i\}$ with $X_i = \{x_{i,j}\}_{j \in [n]}$, there exists a $(\log n)$-degree abecedarian polynomial $f \in \mathbb{F}\langle \mathbf{x} \rangle$ such that

Tight Separation in a Structured Setting

$\{X_1, \ldots, X_m\}$: Partition of the underlying set of variables $\{x_1, \ldots, x_n\}$.

Ordered Set-Multilinear Polynomials: Every monomial has the form $X_1 X_2 \cdots X_m$.

Abecedarian Polynomials: Every monomial has the form $X_1^* X_2^* \cdots X_m^*$.

Abecedarian Formulas: Every gate can be labelled by bucket indices of the end points.

[**Cha 21**]: For $\mathbf{x} = \cup_{i \in [n]} \{X_i\}$ with $X_i = \{x_{i,j}\}_{j \in [n]}$, there exists a $(\log n)$-degree abecedarian polynomial $f \in \mathbb{F}\langle \mathbf{x} \rangle$ such that

- There is an abecedarian ABP of size $O(nd)$ that computes f.

Tight Separation in a Structured Setting

$\{X_1, \ldots, X_m\}$: Partition of the underlying set of variables $\{x_1, \ldots, x_n\}$.

Ordered Set-Multilinear Polynomials: Every monomial has the form $X_1 X_2 \cdots X_m$.

Abecedarian Polynomials: Every monomial has the form $X_1^* X_2^* \cdots X_m^*$.

Abecedarian Formulas: Every gate can be labelled by bucket indices of the end points.

[**Cha 21**]: For $\mathbf{x} = \cup_{i \in [n]} \{X_i\}$ with $X_i = \{x_{i,j}\}_{j \in [n]}$, there exists a $(\log n)$-degree abecedarian polynomial $f \in \mathbb{F}\langle \mathbf{x} \rangle$ such that

- There is an abecedarian ABP of size $O(nd)$ that computes f.
- Any abecedarian formula computing f has size $n^{\Omega(\log \log n)}$.

Tight Separation in a Structured Setting

$\{X_1, \ldots, X_m\}$: Partition of the underlying set of variables $\{x_1, \ldots, x_n\}$.

Ordered Set-Multilinear Polynomials: Every monomial has the form $X_1 X_2 \cdots X_m$.

Abecedarian Polynomials: Every monomial has the form $X_1^* X_2^* \cdots X_m^*$.

Abecedarian Formulas: Every gate can be labelled by bucket indices of the end points.

[**Cha 21**]: For $\mathbf{x} = \cup_{i \in [n]} \{X_i\}$ with $X_i = \{x_{i,j}\}_{j \in [n]}$, there exists a $(\log n)$-degree abecedarian polynomial $f \in \mathbb{F}\langle \mathbf{x} \rangle$ such that

- There is an abecedarian ABP of size $O(nd)$ that computes f.
- Any abecedarian formula computing f has size $n^{\Omega(\log \log n)}$.
- There is an abecedarian formula of size $n^{O(\log \log n)}$ that computes f.

Tight Separation in a Structured Setting

$\{X_1, \ldots, X_m\}$: Partition of the underlying set of variables $\{x_1, \ldots, x_n\}$.

Ordered Set-Multilinear Polynomials: Every monomial has the form $X_1 X_2 \cdots X_m$.

Abecedarian Polynomials: Every monomial has the form $X_1^* X_2^* \cdots X_m^*$.

Abecedarian Formulas: Every gate can be labelled by bucket indices of the end points.

[**Cha 21**]: For $\mathbf{x} = \cup_{i \in [n]} \{X_i\}$ with $X_i = \{x_{i,j}\}_{j \in [n]}$, there exists a $(\log n)$-degree abecedarian polynomial $f \in \mathbb{F}\langle \mathbf{x} \rangle$ such that

- There is an abecedarian ABP of size $O(nd)$ that computes f.
- Any abecedarian formula computing f has size $n^{\Omega(\log \log n)}$.
- There is an abecedarian formula of size $n^{O(\log \log n)}$ that computes f.

If an n-variate polynomial is abecedarian with respect to $\{X_1, \ldots, X_m\}$ for $m = \log n$,

Tight Separation in a Structured Setting

$\{X_1, \ldots, X_m\}$: Partition of the underlying set of variables $\{x_1, \ldots, x_n\}$.

Ordered Set-Multilinear Polynomials: Every monomial has the form $X_1 X_2 \cdots X_m$.

Abecedarian Polynomials: Every monomial has the form $X_1^* X_2^* \cdots X_m^*$.

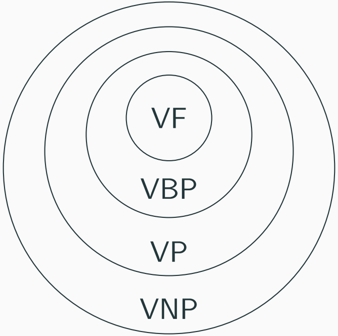
Abecedarian Formulas: Every gate can be labelled by bucket indices of the end points.

[**Cha 21**]: For $\mathbf{x} = \cup_{i \in [n]} \{X_i\}$ with $X_i = \{x_{i,j}\}_{j \in [n]}$, there exists a $(\log n)$-degree abecedarian polynomial $f \in \mathbb{F}\langle \mathbf{x} \rangle$ such that

- There is an abecedarian ABP of size $O(nd)$ that computes f.
- Any abecedarian formula computing f has size $n^{\Omega(\log \log n)}$.
- There is an abecedarian formula of size $n^{O(\log \log n)}$ that computes f.

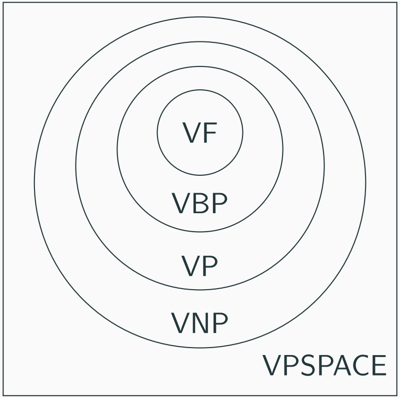
If an n-variate polynomial is abecedarian with respect to $\{X_1, \ldots, X_m\}$ for $m = \log n$, then any formula computing f can be made abecedarian with only $\text{poly}(n)$ blow-up in size.

Classes Beyond VNP



Classes Beyond VNP

$\mathrm{VPSPACE}_b$: Polynomials whose coefficients can be computed in PSPACE/ poly and have degree bounded by poly(n).

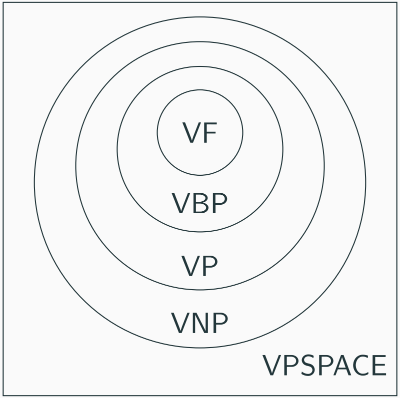


Classes Beyond VNP

$\mathsf{VPSPACE}_b$: Polynomials whose coefficients can be computed in $\mathsf{PSPACE/poly}$ and have degree bounded by $\mathrm{poly}(n)$.

[Koiran-Perifel 09]

$$\mathsf{VNP} \neq \mathsf{VPSPACE}_b \implies \mathsf{P/poly} \neq \mathsf{PSPACE/poly}.$$



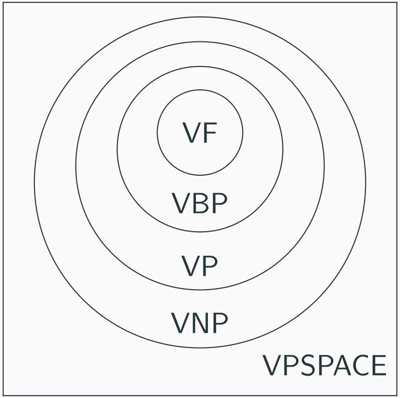
Classes Beyond VNP

$\mathsf{VPSPACE}_b$: Polynomials whose coefficients can be computed in PSPACE/ poly and have degree bounded by poly(n).

[Koiran-Perifel 09]

$$\mathsf{VNP} \neq \mathsf{VPSPACE}_b \implies \mathsf{P/poly} \neq \mathsf{PSPACE/poly}.$$

$$\boxed{\mathsf{VNP} \stackrel{?}{=} \mathsf{VPSPACE}_b}$$



Classes Beyond VNP

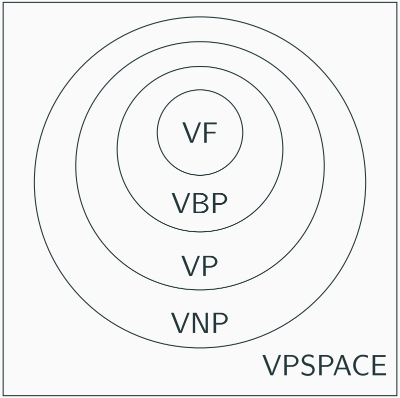
$\mathsf{VPSPACE}_b$: Polynomials whose coefficients can be computed in PSPACE/ poly and have degree bounded by $\mathrm{poly}(n)$.

[Koiran-Perifel 09]

$$\mathsf{VNP} \neq \mathsf{VPSPACE}_b \implies \mathsf{P}/\mathrm{poly} \neq \mathsf{PSPACE}/\mathrm{poly}.$$

$$\boxed{\mathsf{VNP} \stackrel{?}{=} \mathsf{VPSPACE}_b}$$

[C-Gajjar-Tengse 23]: $\mathsf{VNP} \neq \mathsf{VPSPACE}_b$ in the monotone setting.



Super-Polynomial Lower Bound against $\sum$osmABPs

An ABP is σ-ordered set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

- there are d layers in the ABP
- every edge in layer i is labelled by a homogeneous linear form in $\mathbf{x}_{\sigma(i)}$

Super-Polynomial Lower Bound against $\sum$osmABPs

An ABP is σ-ordered set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

- there are d layers in the ABP
- every edge in layer i is labelled by a homogeneous linear form in $\mathbf{x}_{\sigma(i)}$

$\sum$osmABP: Sum of ordered set-multilinear ABPs, each with a possibly different ordering.

Super-Polynomial Lower Bound against $\sum$osmABPs

An ABP is σ-ordered set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

- there are d layers in the ABP
- every edge in layer i is labelled by a homogeneous linear form in $\mathbf{x}_{\sigma(i)}$

$\sum$osmABP: Sum of ordered set-multilinear ABPs, each with a possibly different ordering.

[C-Kush-Saraf-Shpilka 24]: For $\omega(\log n) = d \leq n$, there is a polynomial $G_{n,d}(\mathbf{x})$ which is set-multilinear w.r.t $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$, where $|\mathbf{x}_i| \leq n$ for every $i \in [d]$, such that:

Super-Polynomial Lower Bound against $\sum$osmABPs

An ABP is σ-ordered set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

- there are d layers in the ABP
- every edge in layer i is labelled by a homogeneous linear form in $\mathbf{x}_{\sigma(i)}$

$\sum$osmABP: Sum of ordered set-multilinear ABPs, each with a possibly different ordering.

[C-Kush-Saraf-Shpilka 24]: For $\omega(\log n) = d \leq n$, there is a polynomial $G_{n,d}(\mathbf{x})$ which is set-multilinear w.r.t $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$, where $|\mathbf{x}_i| \leq n$ for every $i \in [d]$, such that:

- $G_{n,d}$ is computable by a set-multilinear ABP of size $\mathrm{poly}(n, d)$,

Super-Polynomial Lower Bound against $\sum$osmABPs

An ABP is σ-ordered set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

- there are d layers in the ABP
- every edge in layer i is labelled by a homogeneous linear form in $\mathbf{x}_{\sigma(i)}$

$\sum$osmABP: Sum of ordered set-multilinear ABPs, each with a possibly different ordering.

[C-Kush-Saraf-Shpilka 24]: For $\omega(\log n) = d \leq n$, there is a polynomial $G_{n,d}(\mathbf{x})$ which is set-multilinear w.r.t $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$, where $|\mathbf{x}_i| \leq n$ for every $i \in [d]$, such that:

- $G_{n,d}$ is computable by a set-multilinear ABP of size $\text{poly}(n, d)$,
- any $\sum$osmABP of max-width $\text{poly}(n)$ computing $G_{n,d}$ requires total-width $2^{\Omega(d)}$,

Super-Polynomial Lower Bound against $\sum$osmABPs

An ABP is σ-ordered set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

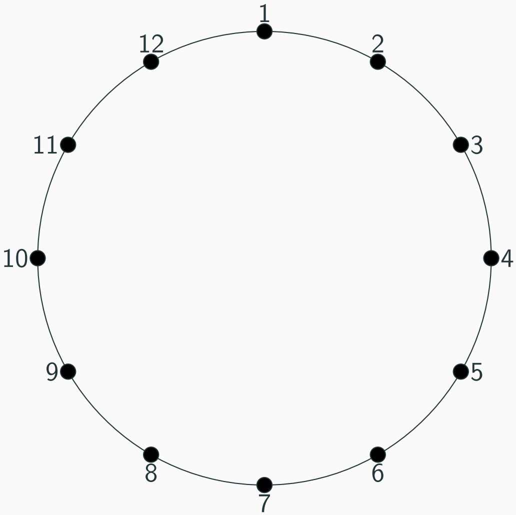
- there are d layers in the ABP
- every edge in layer i is labelled by a homogeneous linear form in $\mathbf{x}_{\sigma(i)}$

$\sum$osmABP: Sum of ordered set-multilinear ABPs, each with a possibly different ordering.

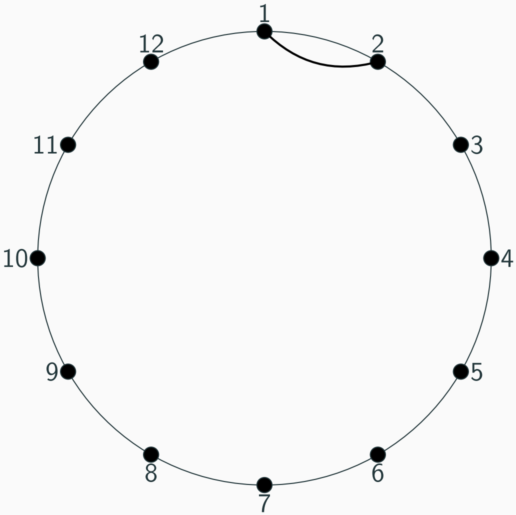
[C-Kush-Saraf-Shpilka 24]: For $\omega(\log n) = d \leq n$, there is a polynomial $G_{n,d}(\mathbf{x})$ which is set-multilinear w.r.t $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$, where $|\mathbf{x}_i| \leq n$ for every $i \in [d]$, such that:

- $G_{n,d}$ is computable by a set-multilinear ABP of size $\text{poly}(n, d)$,
- any $\sum$osmABP of max-width $\text{poly}(n)$ computing $G_{n,d}$ requires total-width $2^{\Omega(d)}$,
- any ordered set-multilinear branching program computing $G_{n,d}$ requires width $n^{\Omega(d)}$.

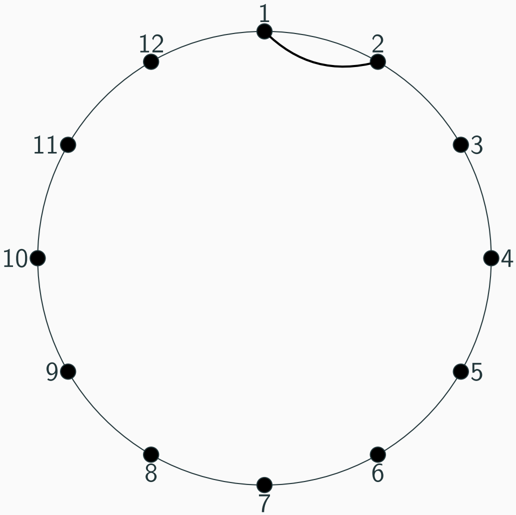
Arc Partition



Arc Partition

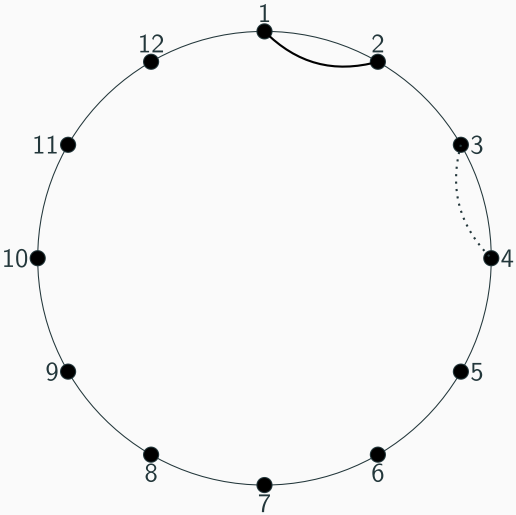


Arc Partition



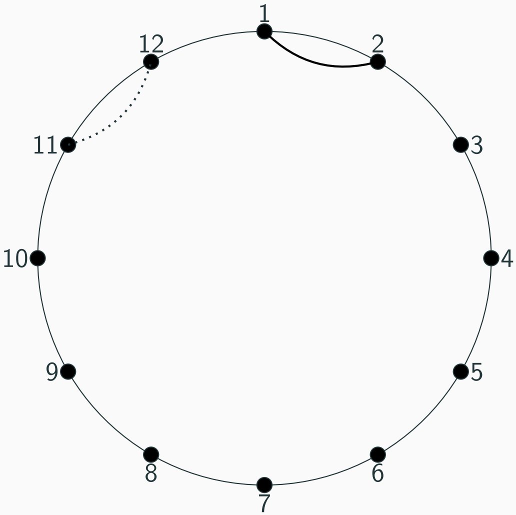
$$\mathcal{P}_1 = \{(1, 2)\}$$

Arc Partition



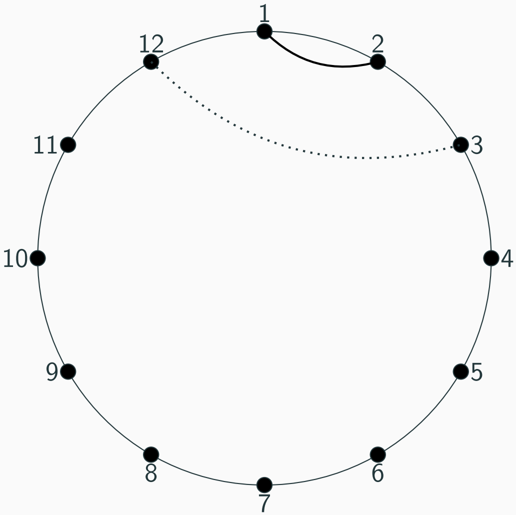
$$\mathcal{P}_1 = \{(1,2)\}$$

Arc Partition



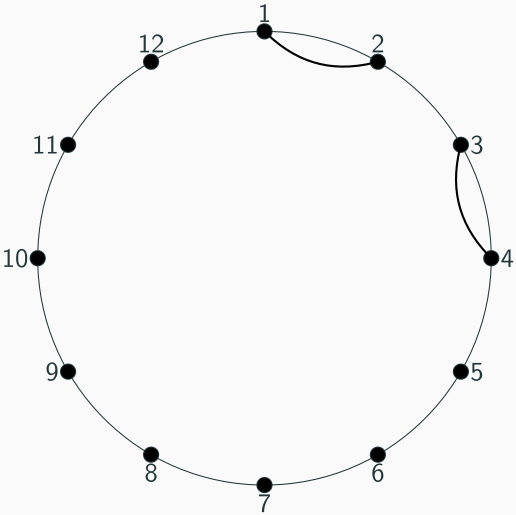
$$\mathcal{P}_1 = \{(1,2)\}$$

Arc Partition



$$\mathcal{P}_1 = \{(1,2)\}$$

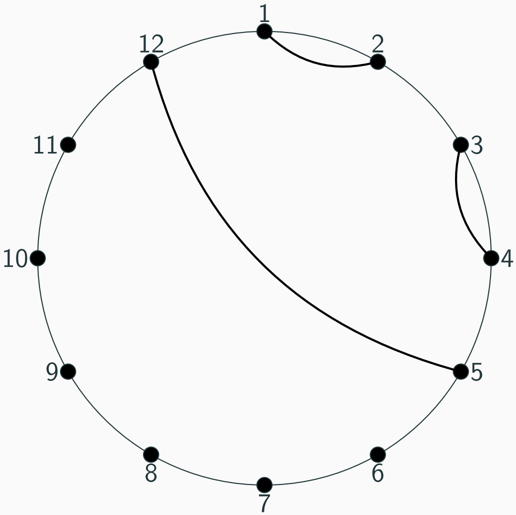
Arc Partition



$$\mathcal{P}_1 = \{(1,2)\}$$

$$\mathcal{P}_2 = \{(1,2),(3,4)\}$$

Arc Partition

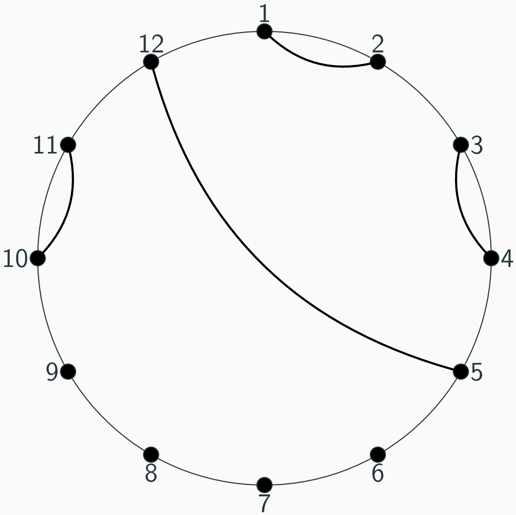


$$\mathcal{P}_1 = \{(1,2)\}$$

$$\mathcal{P}_2 = \{(1,2),(3,4)\}$$

$$\mathcal{P}_3 = \{(1,2),(3,4),(12,5)\}$$

Arc Partition



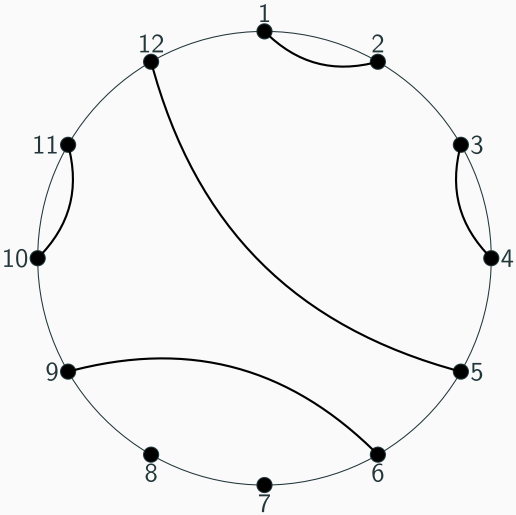
$$\mathcal{P}_1 = \{(1,2)\}$$

$$\mathcal{P}_2 = \{(1,2),(3,4)\}$$

$$\mathcal{P}_3 = \{(1,2),(3,4),(12,5)\}$$

$$\mathcal{P}_4 = \{(1,2),(3,4),(12,5),(10,11)\}$$

Arc Partition



$$\mathcal{P}_1 = \{(1,2)\}$$

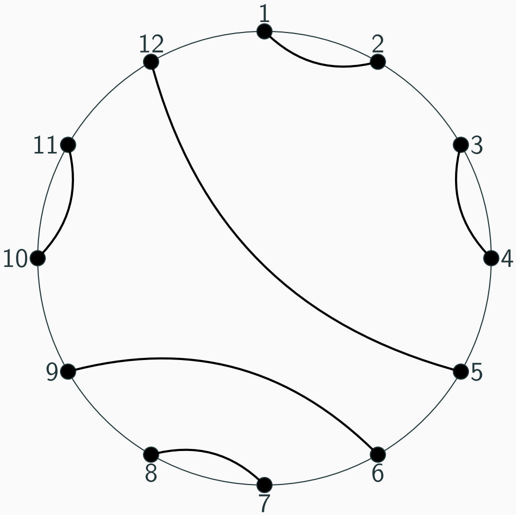
$$\mathcal{P}_2 = \{(1,2),(3,4)\}$$

$$\mathcal{P}_3 = \{(1,2),(3,4),(12,5)\}$$

$$\mathcal{P}_4 = \{(1,2),(3,4),(12,5),(10,11)\}$$

$$\mathcal{P}_5 = \{(1,2),(3,4),(12,5),(10,11),(9,6)\}$$

Arc Partition



$$\mathcal{P}_1 = \{(1,2)\}$$

$$\mathcal{P}_2 = \{(1,2),(3,4)\}$$

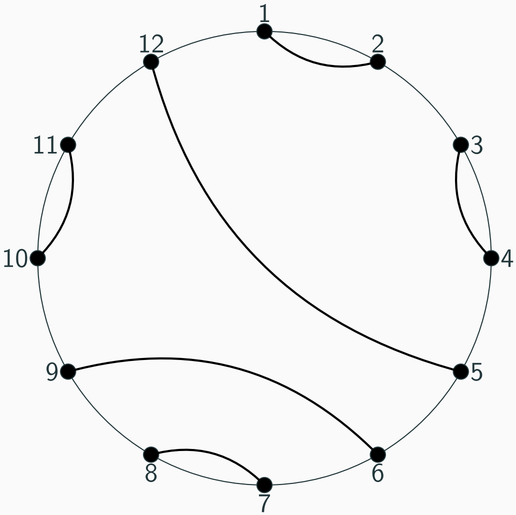
$$\mathcal{P}_3 = \{(1,2),(3,4),(12,5)\}$$

$$\mathcal{P}_4 = \{(1,2),(3,4),(12,5),(10,11)\}$$

$$\mathcal{P}_5 = \{(1,2),(3,4),(12,5),(10,11),(9,6)\}$$

$$\mathcal{P}_6 = \{(1,2),(3,4),(12,5),(10,11),(9,6),(8,7)\}$$

Arc Partition



$$\mathcal{P}_1 = \{(1,2)\}$$

$$\mathcal{P}_2 = \{(1,2),(3,4)\}$$

$$\mathcal{P}_3 = \{(1,2),(3,4),(12,5)\}$$

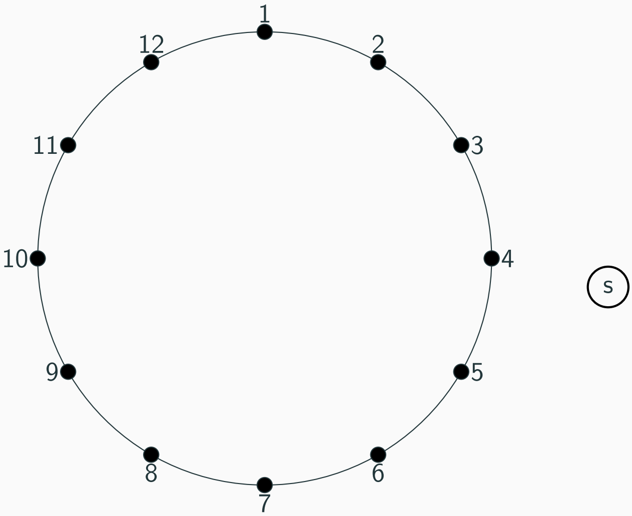
$$\mathcal{P}_4 = \{(1,2),(3,4),(12,5),(10,11)\}$$

$$\mathcal{P}_5 = \{(1,2),(3,4),(12,5),(10,11),(9,6)\}$$

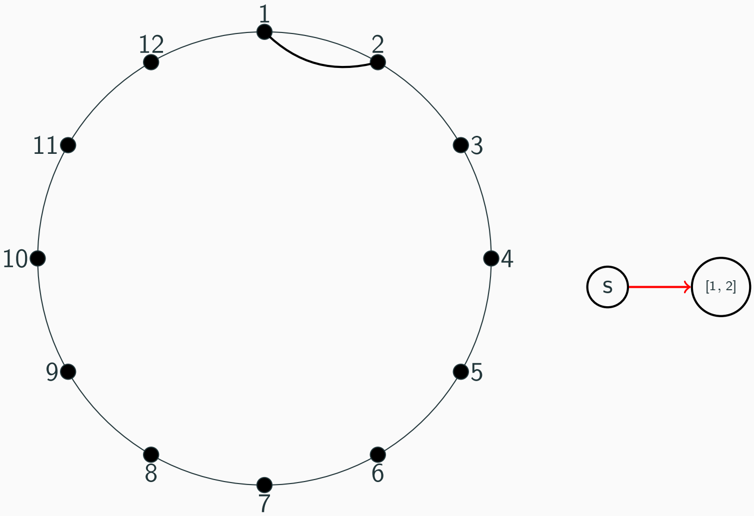
$$\mathcal{P}_6 = \{(1,2),(3,4),(12,5),(10,11),(9,6),(8,7)\}$$

$\mathbf{P}_6$ = All possibles sequences of such pairs.

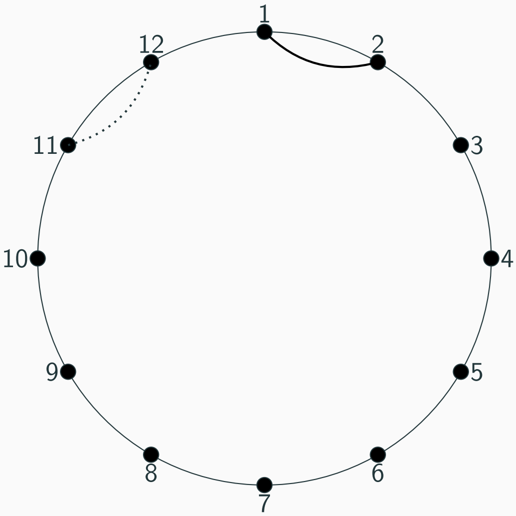
The ABP Upper Bound

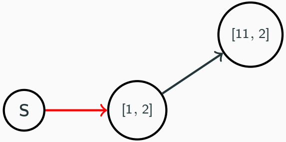


The ABP Upper Bound

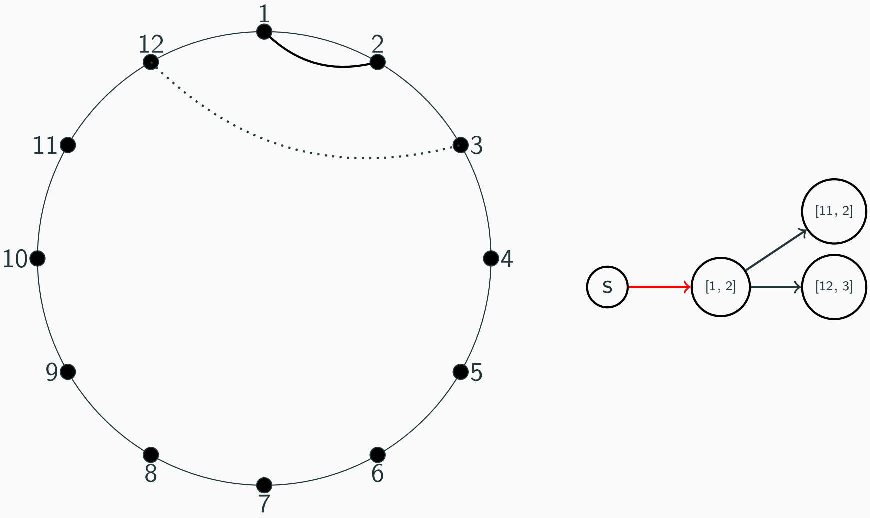


The ABP Upper Bound

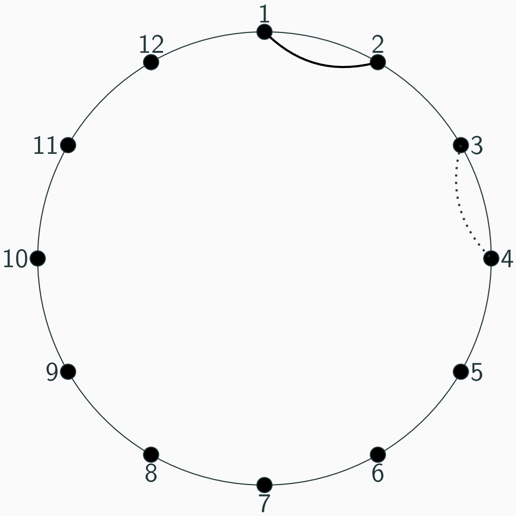


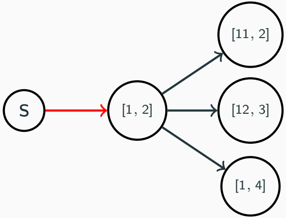


The ABP Upper Bound

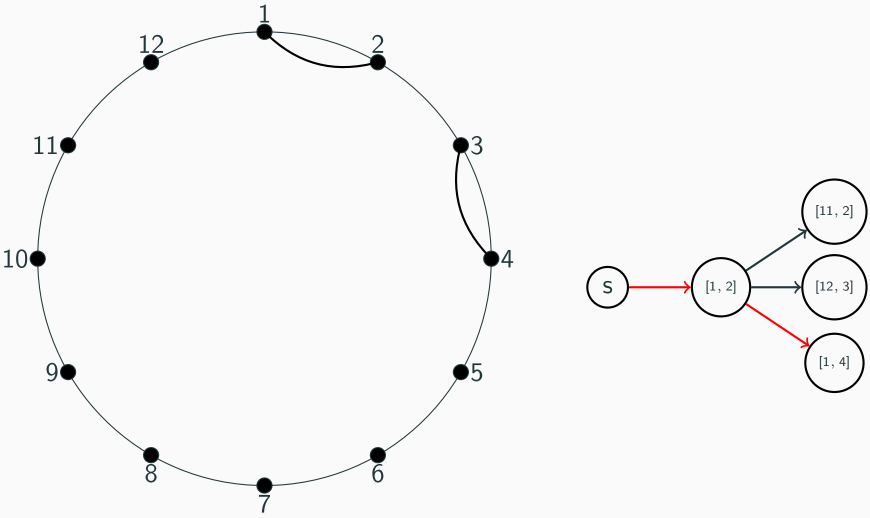


The ABP Upper Bound

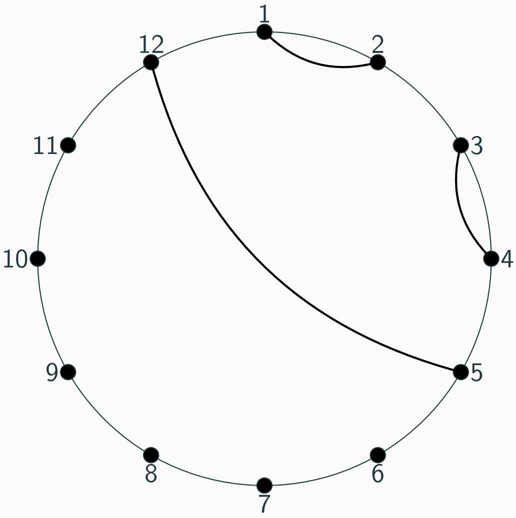


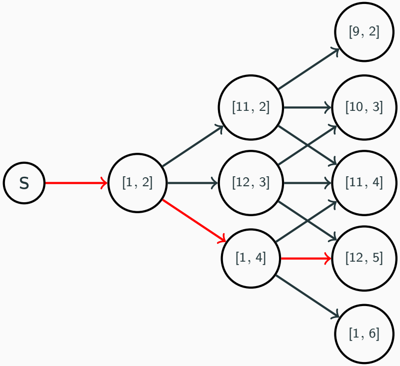


The ABP Upper Bound

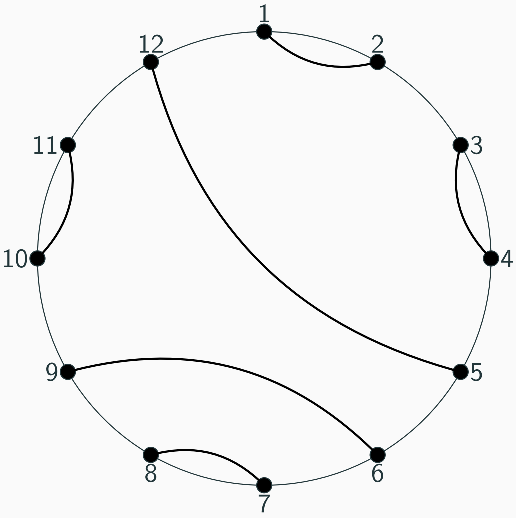


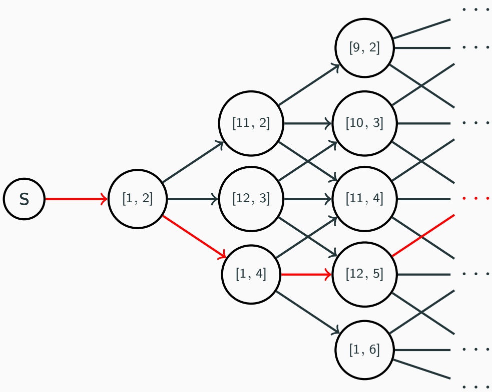
The ABP Upper Bound



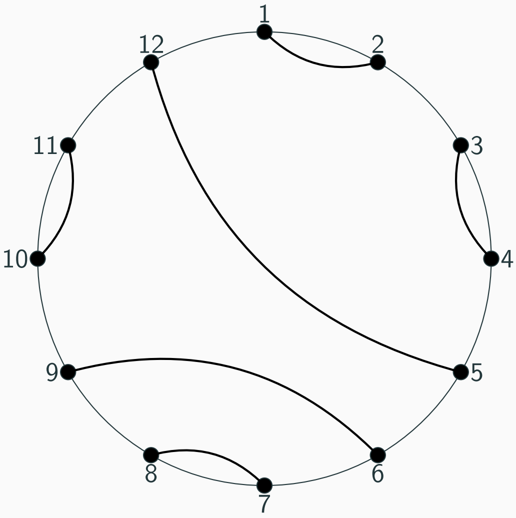


The ABP Upper Bound

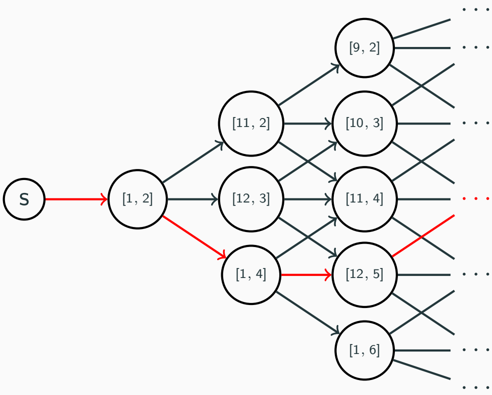




The ABP Upper Bound



Every path corresponds to an element in $\mathbf{P}_{d/2}$.

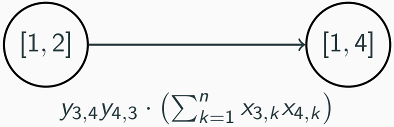


The Hard Polynomial

The Hard Polynomial

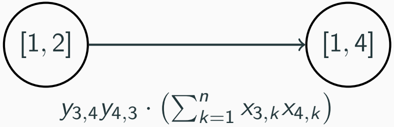
The new pair: $(3, 4)$.

The Hard Polynomial



The new pair: $(3, 4)$.

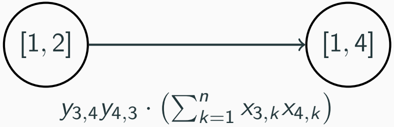
The Hard Polynomial



The new pair: $(3,4)$.

$(y_{3,4}y_{4,3})$: To select.

The Hard Polynomial

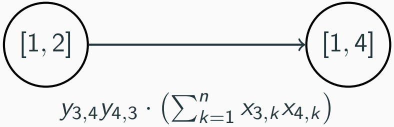


The new pair: $(3,4)$.

$(y_{3,4}y_{4,3})$: To select.

$\left(\sum_{k=1}^{n} x_{3,k}x_{4,k}\right)$: To achieve full-rank.

The Hard Polynomial



The new pair: $(3, 4)$.

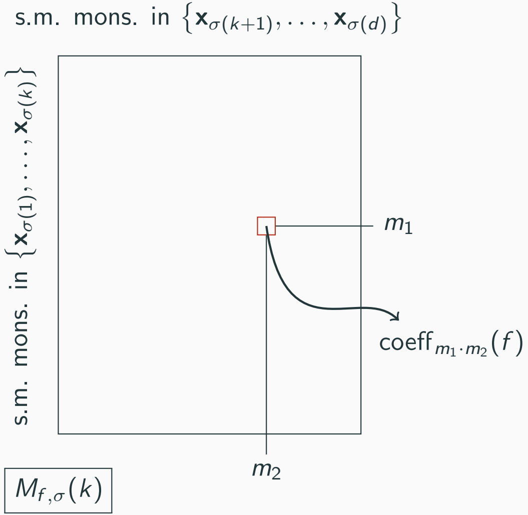
$(y_{3,4}y_{4,3})$: To select.

$\left(\sum_{k=1}^{n} x_{3,k}x_{4,k}\right)$: To achieve full-rank.

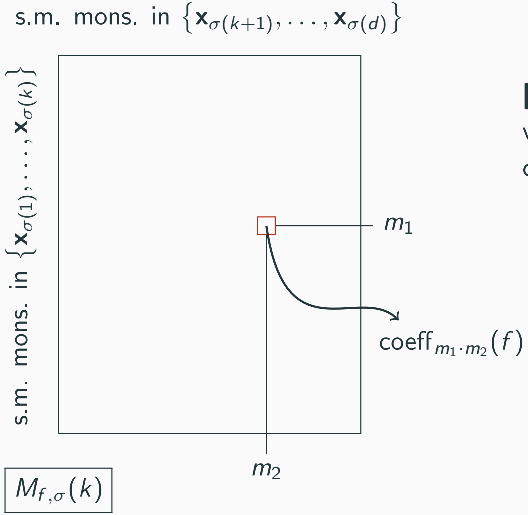
	$x_{4,1}$	$x_{4,2}$	$\cdots$	$\cdots$	$x_{4,n}$
$x_{3,1}$	1	0	$\cdots$	$\cdots$	0
$x_{3,2}$	0	1	$\cdots$	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$			$\vdots$
$\vdots$	$\vdots$	$\vdots$			$\vdots$
$x_{3,n}$	0	0	$\cdots$	$\cdots$	1

Lower Bound for a single osmABP

f is a set-multilinear poly. w.r.t $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$.



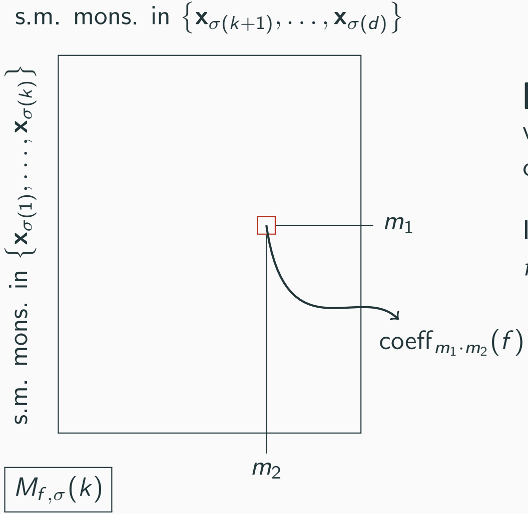
Lower Bound for a single osmABP



f is a set-multilinear poly. w.r.t $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$.

[Nisan 91]: For every $1 \leq k \leq d$, the number of vertices in the k-th layer of the smallest osmABP(σ) computing f is equal to the rank of $M_{f,\sigma}(k)$.

Lower Bound for a single osmABP



f is a set-multilinear poly. w.r.t $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$.

[Nisan 91]: For every $1 \leq k \leq d$, the number of vertices in the k-th layer of the smallest osmABP(σ) computing f is equal to the rank of $M_{f,\sigma}(k)$.

If $\mathcal{A}$ is the smallest osmABP (in order σ) computing f, then

$$\mathrm{size}(\mathcal{A}) = \sum_{i=1}^{d} \mathrm{rank}(M_{f,\sigma}(k)).$$

Lower Bound for a single osmABP (contd.)

$$G_{n,d} = \sum_{\mathcal{P} \in \mathbf{P}_{d/2}} \prod_{(i,j) \in \mathcal{P}} y_{i,j} y_{j,i} \cdot \left(\sum_{k=1}^{n} x_{i,k} x_{j,k} \right).$$

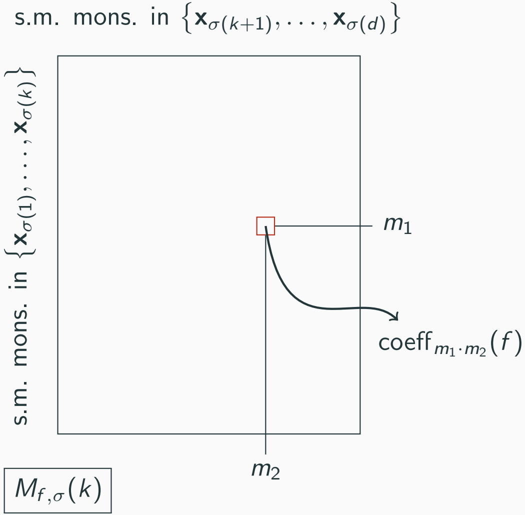
Lower Bound for a single osmABP (contd.)

$$G_{n,d} = \sum_{\mathcal{P} \in \mathbf{P}_{d/2}} \prod_{(i,j) \in \mathcal{P}} y_{i,j} y_{j,i} \cdot \left(\sum_{k=1}^{n} x_{i,k} x_{j,k} \right).$$

Properties:

- $G_{n,d}$ is computable by a set-multilinear ABP of size poly(n, d).

Lower Bound for a single osmABP (contd.)

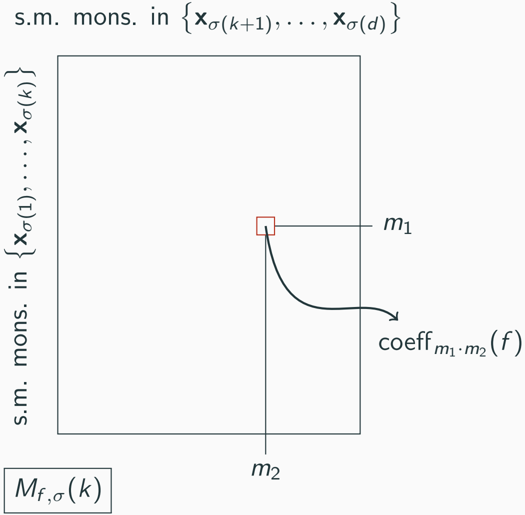


$$G_{n,d} = \sum_{\mathcal{P} \in \mathbf{P}_{d/2}} \prod_{(i,j) \in \mathcal{P}} y_{i,j} y_{j,i} \cdot \left(\sum_{k=1}^{n} x_{i,k} x_{j,k} \right).$$

Properties:

- $G_{n,d}$ is computable by a set-multilinear ABP of size poly(n, d).
- For every $\sigma \in S_d$, there is some $\mathcal{P}$ such that for at least $d/8$ of the $P = (i,j) \in \mathcal{P}$, $i \in \{\sigma(1), \ldots \sigma(\frac{d}{2})\}$ & $j \in \{\sigma(1 + \frac{d}{2})), \ldots \sigma(d)\}$.

Lower Bound for a single osmABP (contd.)



$$G_{n,d} = \sum_{\mathcal{P} \in \mathbf{P}_{d/2}} \prod_{(i,j) \in \mathcal{P}} y_{i,j} y_{j,i} \cdot \left(\sum_{k=1}^{n} x_{i,k} x_{j,k} \right).$$

Properties:

- $G_{n,d}$ is computable by a set-multilinear ABP of size $\mathrm{poly}(n,d)$.
- For every $\sigma \in S_d$, there is some $\mathcal{P}$ such that for at least $d/8$ of the $P = (i,j) \in \mathcal{P}$, $i \in \{\sigma(1), \ldots \sigma(\frac{d}{2})\}$ & $j \in \{\sigma(1 + \frac{d}{2})), \ldots \sigma(d)\}$.

  Therefore,

$$\mathrm{rank}(M_{G_{n,d},\sigma}(d/2)) = \Omega(n^{d/8}).$$

Lower Bound for a Sum of osmABPs

- $\{M_w(f) \ : \ w \in \mathcal{S}\}$ is a set of matrices such that $M_w(G_{n,d})$ has full rank for every $w \in \mathcal{S}$.

Lower Bound for a Sum of osmABPs

- $\{M_w(f) \;:\; w \in \mathcal{S}\}$ is a set of matrices such that $M_w(G_{n,d})$ has full rank for every $w \in \mathcal{S}$.
- If $G_{n,d}$ is computed by a sum of t osmABPs, then

$$G_{n,d} = \sum_{i=1}^{t} g_i \quad \text{where} \quad g_i = \sum_{u_1,\ldots,u_{q-1}} \prod_{j=1}^{q} g^{(i)}_{u_{j-1},u_j}.$$

Lower Bound for a Sum of osmABPs

- $\{M_w(f) \ : \ w \in \mathcal{S}\}$ is a set of matrices such that $M_w(G_{n,d})$ has full rank for every $w \in \mathcal{S}$.
- If $G_{n,d}$ is computed by a sum of t osmABPs, then

$$G_{n,d} = \sum_{i=1}^{t} g_i \quad \text{where} \quad g_i = \sum_{u_1,\ldots,u_{q-1}} \prod_{j=1}^{q} g^{(i)}_{u_{j-1},u_j}.$$

- Define a distribution $\mathcal{D}$ on $\mathcal{S}$ such that when $w \sim \mathcal{D}$, if g_is are computable by osmABPs efficiently, then

Lower Bound for a Sum of osmABPs

- $\{M_w(f) \;:\; w \in \mathcal{S}\}$ is a set of matrices such that $M_w(G_{n,d})$ has full rank for every $w \in \mathcal{S}$.
- If $G_{n,d}$ is computed by a sum of t osmABPs, then

$$G_{n,d} = \sum_{i=1}^{t} g_i \quad \text{where} \quad g_i = \sum_{u_1,\ldots,u_{q-1}} \prod_{j=1}^{q} g^{(i)}_{u_{j-1},u_j}.$$

- Define a distribution $\mathcal{D}$ on $\mathcal{S}$ such that when $w \sim \mathcal{D}$, if g_is are computable by osmABPs efficiently, then

  for every i, w.h.p. there are many js, for which $M_w(g^{(i)}_{u_{j-1},u_j})$ is far from full rank

Lower Bound for a Sum of osmABPs

- $\{M_w(f) \,:\, w \in \mathcal{S}\}$ is a set of matrices such that $M_w(G_{n,d})$ has full rank for every $w \in \mathcal{S}$.
- If $G_{n,d}$ is computed by a sum of t osmABPs, then

$$G_{n,d} = \sum_{i=1}^{t} g_i \quad \text{where} \quad g_i = \sum_{u_1,\ldots,u_{q-1}} \prod_{j=1}^{q} g^{(i)}_{u_{j-1},u_j}.$$

- Define a distribution $\mathcal{D}$ on $\mathcal{S}$ such that when $w \sim \mathcal{D}$, if g_is are computable by osmABPs efficiently, then

for every i, w.h.p. there are many js, for which $M_w(g^{(i)}_{u_{j-1},u_j})$ is far from full rank

$\implies$ for every i, w.h.p. $M_w(g_i)$ is far from full rank

Lower Bound for a Sum of osmABPs

- $\{M_w(f) \ : \ w \in \mathcal{S}\}$ is a set of matrices such that $M_w(G_{n,d})$ has full rank for every $w \in \mathcal{S}$.
- If $G_{n,d}$ is computed by a sum of t osmABPs, then

$$G_{n,d} = \sum_{i=1}^{t} g_i \quad \text{where} \quad g_i = \sum_{u_1,\ldots,u_{q-1}} \prod_{j=1}^{q} g^{(i)}_{u_{j-1},u_j}.$$

- Define a distribution $\mathcal{D}$ on $\mathcal{S}$ such that when $w \sim \mathcal{D}$, if g_is are computable by osmABPs efficiently, then

for every i, w.h.p. there are many js, for which $M_w(g^{(i)}_{u_{j-1},u_j})$ is far from full rank

$\implies$ for every i, w.h.p. $M_w(g_i)$ is far from full rank

$\implies$ $M_w(G_{n,d})$ is far from full rank unless t is large.

Some Open Directions

- Better lower bounds against homogeneous formulas?

Some Open Directions

- Better lower bounds against homogeneous formulas?
- Better lower bounds against set-multilinear ABPs?

Some Open Directions

- Better lower bounds against homogeneous formulas?
- Better lower bounds against set-multilinear ABPs?
- PIT for $\sum$ osmABP?

Some Open Directions

- Better lower bounds against homogeneous formulas?
- Better lower bounds against set-multilinear ABPs?
- PIT for $\sum$ osmABP?
- Bootstrapping statement, similar to **[Carmossino-Impagiazzo-Lovett-Mihajlin 18]**, which is sensitive to both degree and number of variables?

Some Open Directions

- Better lower bounds against homogeneous formulas?
- Better lower bounds against set-multilinear ABPs?
- PIT for $\sum$ osmABP?
- Bootstrapping statement, similar to **[Carmossino-Impagiazzo-Lovett-Mihajlin 18]**, which is sensitive to both degree and number of variables?
- Separating formulas and ABPs in the non-commutative setting?

Some Open Directions

- Better lower bounds against homogeneous formulas?
- Better lower bounds against set-multilinear ABPs?
- PIT for $\sum$ osmABP?
- Bootstrapping statement, similar to **[Carmossino-Impagiazzo-Lovett-Mihajlin 18]**, which is sensitive to both degree and number of variables?
- Separating formulas and ABPs in the non-commutative setting?
- Meaningful definition of VPH?

Some Open Directions

- Better lower bounds against homogeneous formulas?
- Better lower bounds against set-multilinear ABPs?
- PIT for $\sum$ osmABP?
- Bootstrapping statement, similar to **[Carmossino-Impagiazzo-Lovett-Mihajlin 18]**, which is sensitive to both degree and number of variables?
- Separating formulas and ABPs in the non-commutative setting?
- Meaningful definition of VPH?

Questions?