Lower Bounds for some Algebraic Models of Computation

Prerona Chatterjee

May 6, 2024

Algebraic Circuit Complexity

Q: Given $f(\mathbf{x}) \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d, how many $+, \times, -$ gates are needed to compute f?

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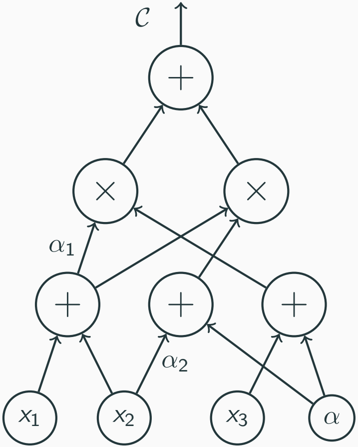

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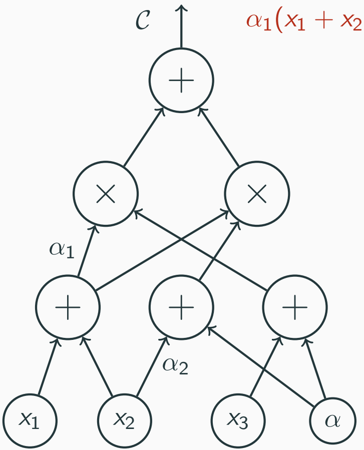

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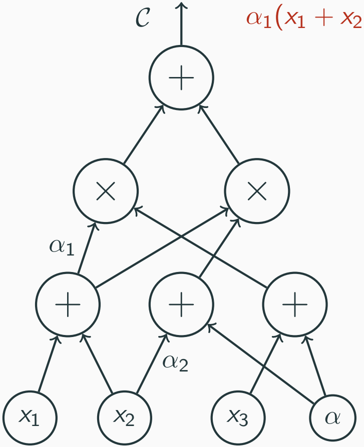

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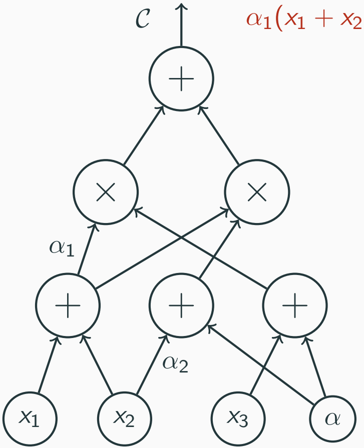

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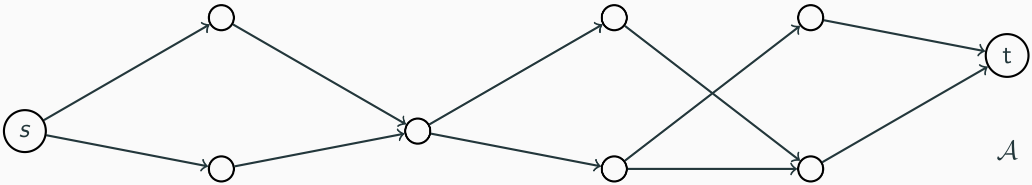
$$\mathsf{VP} = \mathsf{VNP} \overset{\text{G.R.H.}}{\Longrightarrow} \mathsf{P} = \mathsf{NP}$$

Algebraic Branching Programs



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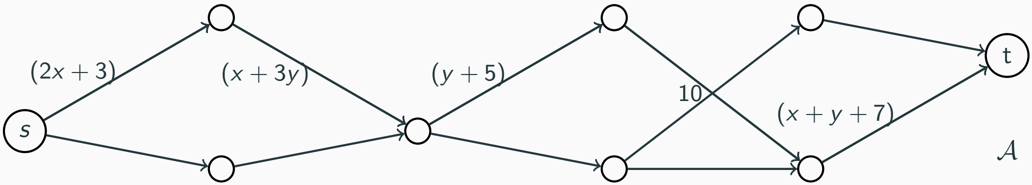

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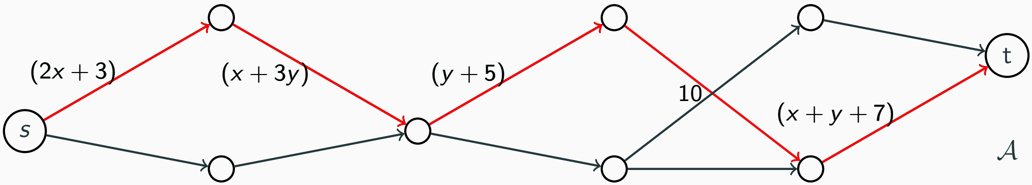

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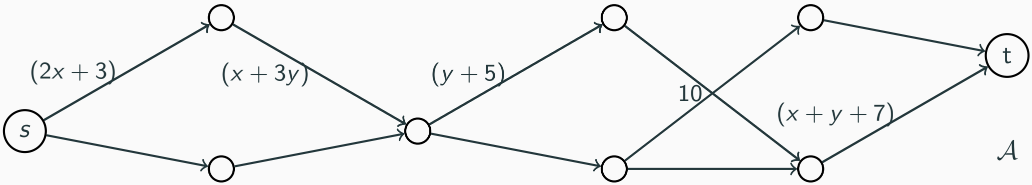

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- Polynomial computed by the ABP: $f_{\mathcal{A}}(\mathbf{x}) = \sum_p \mathrm{wt}(p)$

Towards Better ABP Lower Bounds

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[C-Kush-Saraf-Shpilka 24]: For $\omega(\log n) = d \leq n$, there is a polynomial $G_{n,d}(\mathbf{x})$ which is set-multilinear w.r.t $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$, where $|\mathbf{x}_i| \leq n$ for every $i \in [d]$, such that:

- $G_{n,d}$ is computable by a set-multilinear ABP of size $\text{poly}(n)$,
- any $\sum$ osmABP computing $G_{n,d}$ must have super-polynomial total-width.

Set-Multilinearity

The variable set is divided into buckets.

$$\mathbf{x} = \mathbf{x}_1 \cup \cdots \cup \mathbf{x}_d \quad \text{where} \quad \mathbf{x}_i = \{x_{i,1}, \ldots x_{i,n_i}\}.$$

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An ABP is set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if every path in it

computes a set-multilinear monomial with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$.

Super-Polynomial Lower Bound against $\sum$osmABPs

For $\sigma \in S_d$, an ABP is σ-ordered set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

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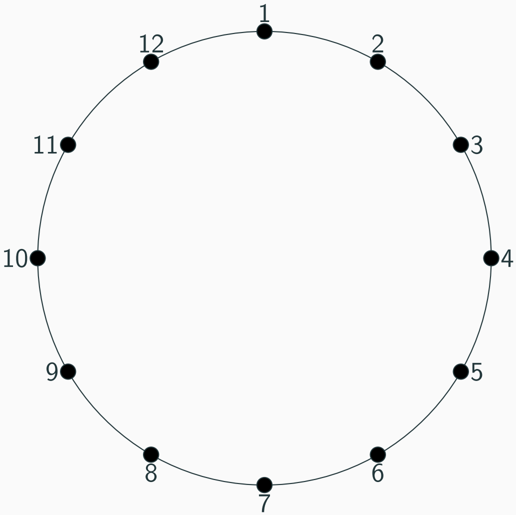
- any ordered set-multilinear branching program computing $G_{n,d}$ requires width $n^{\Omega(d)}$.

Proof Ideas

Arc Partition



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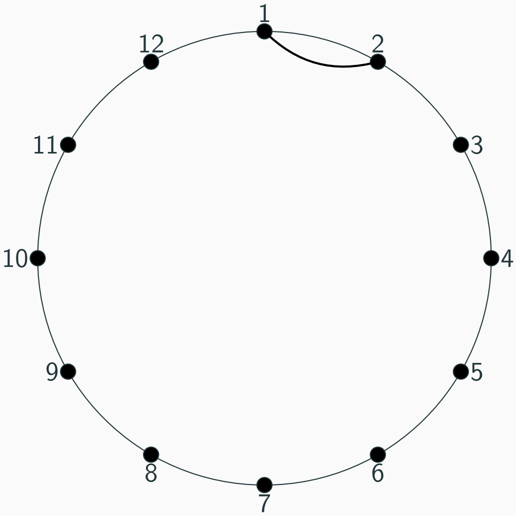

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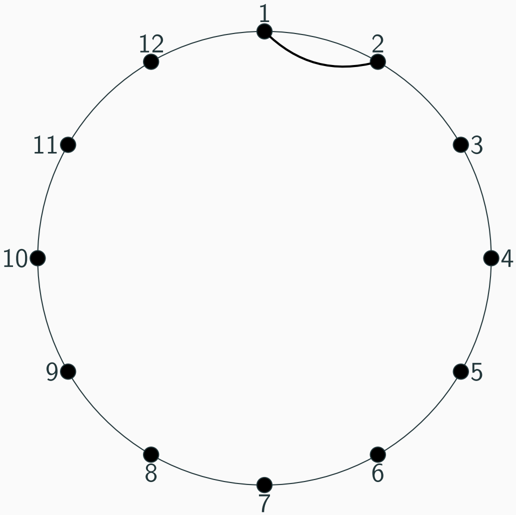

$\mathcal{P}_1 = \{(1, 2)\}$

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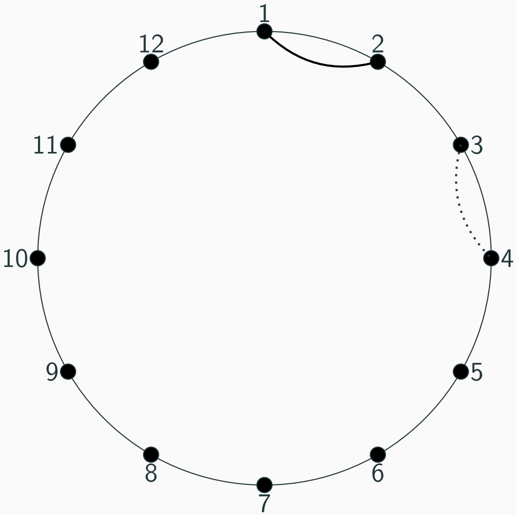

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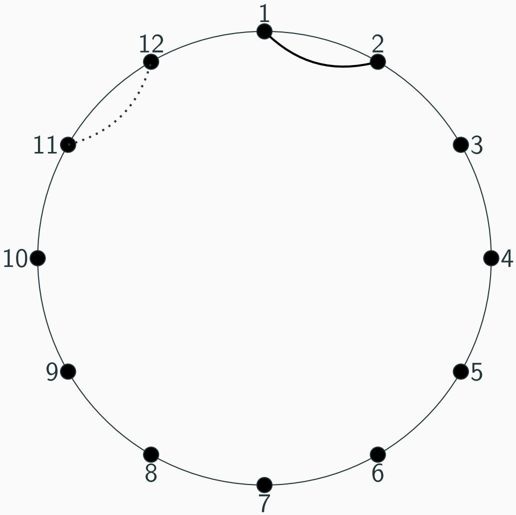

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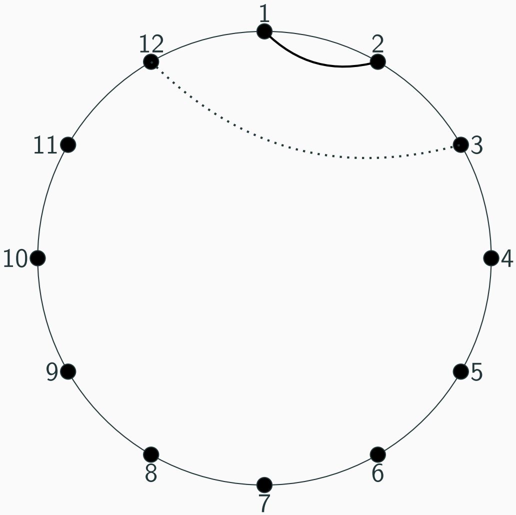

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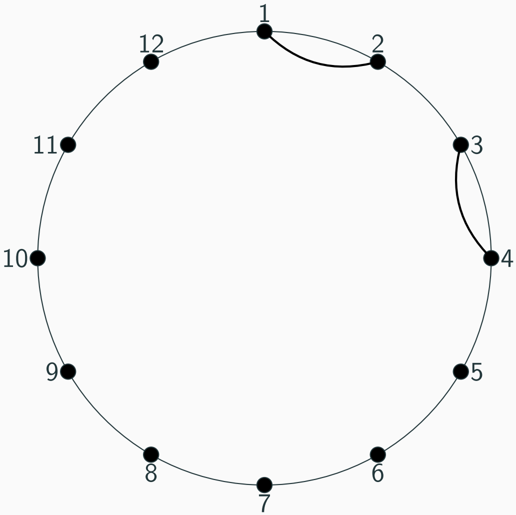

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$$\mathcal{P}_2 = \{(1,2),(3,4)\}$$

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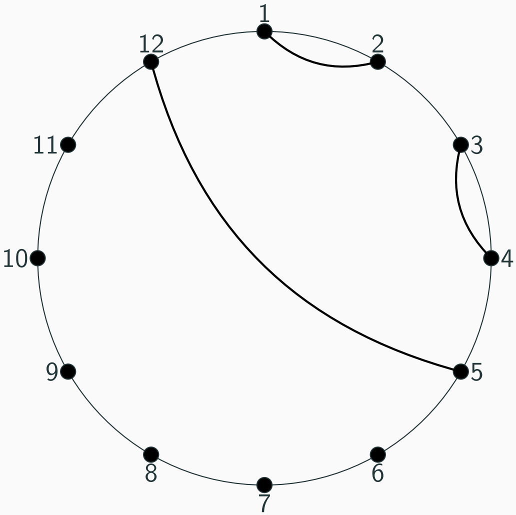

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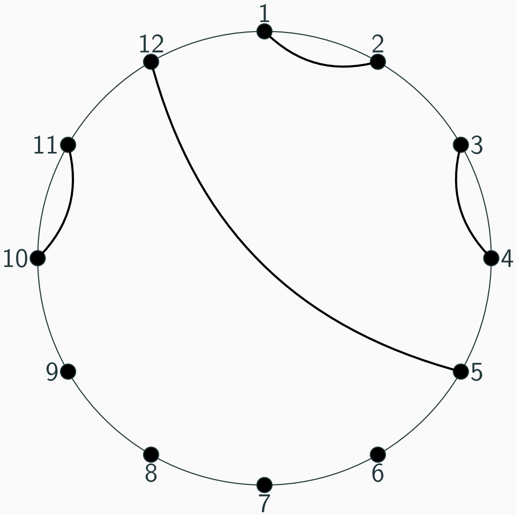

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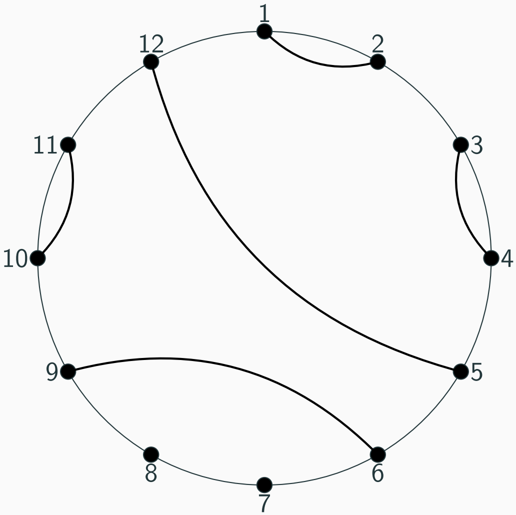

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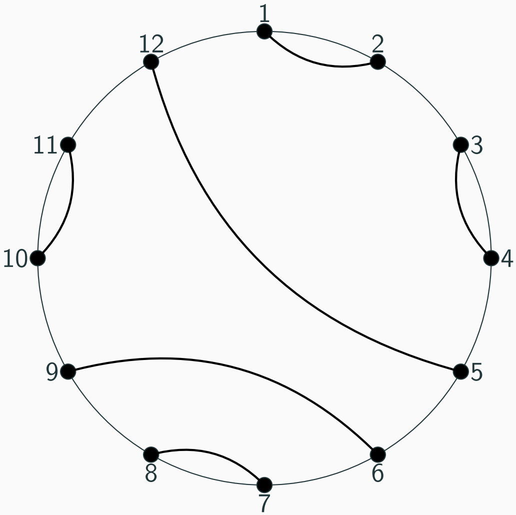

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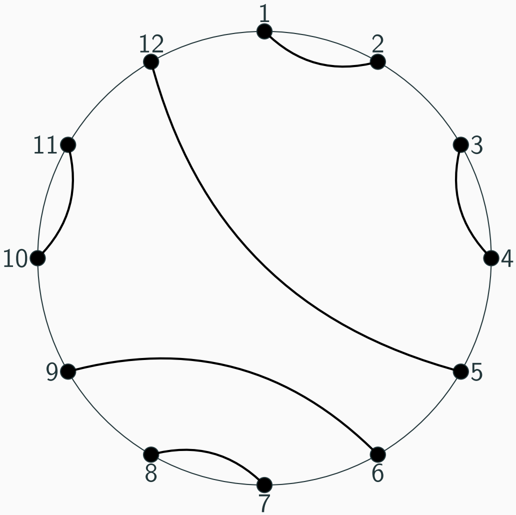

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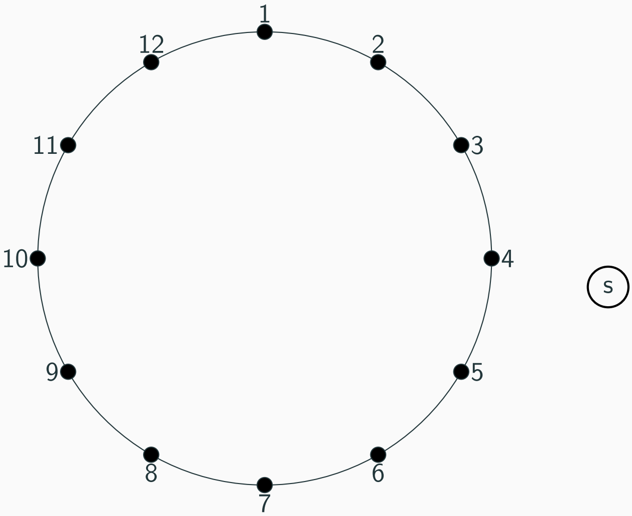
$\mathbf{P}_6$ = All possibles sequences of such pairs.

The ABP Upper Bound



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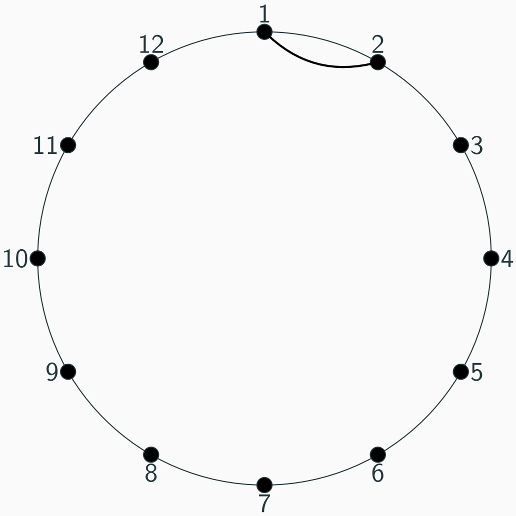

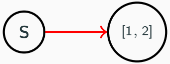

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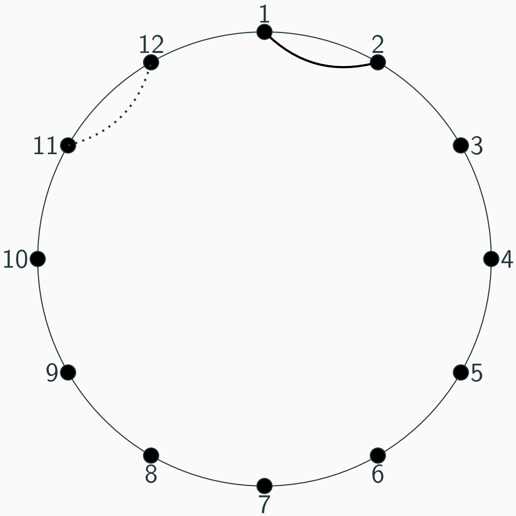

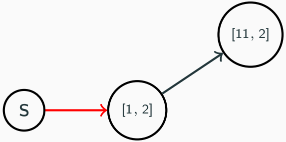

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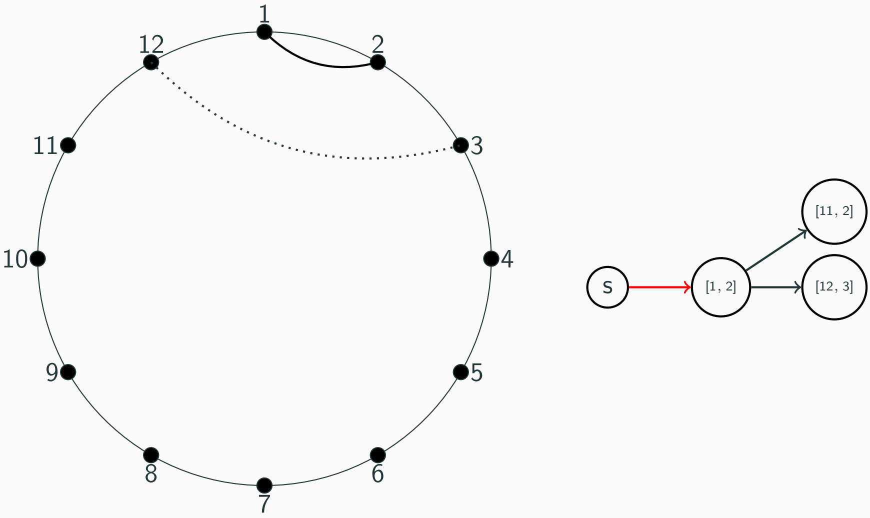

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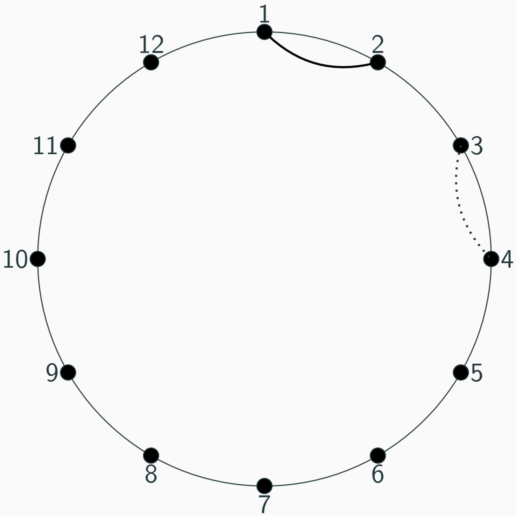

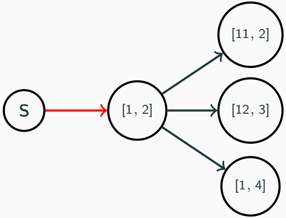

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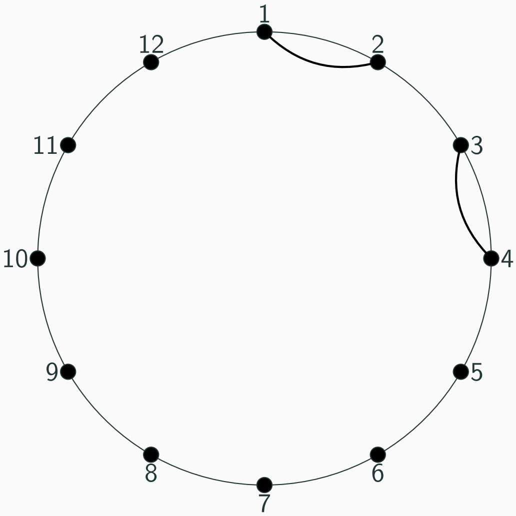

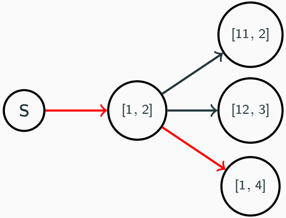

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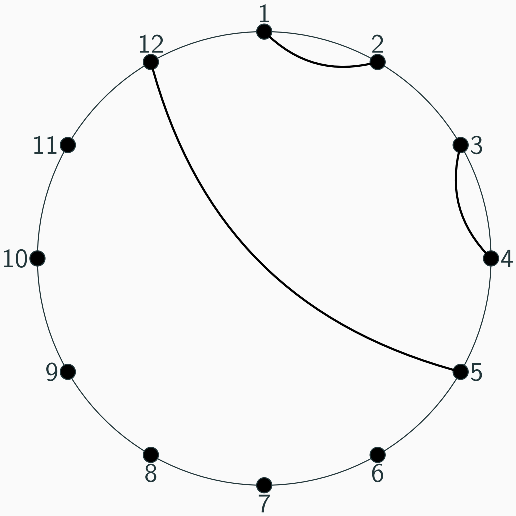

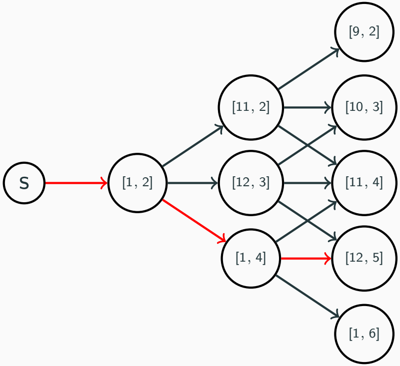

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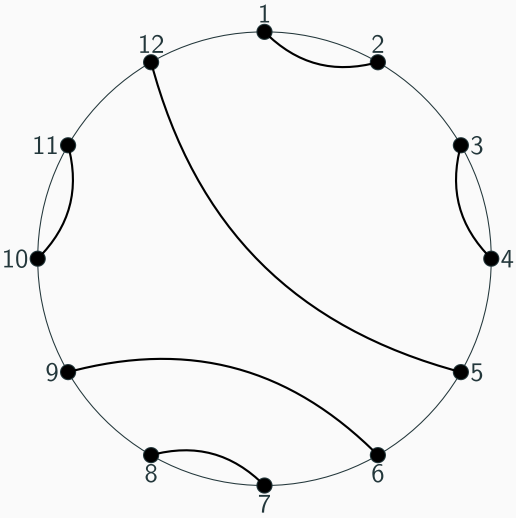

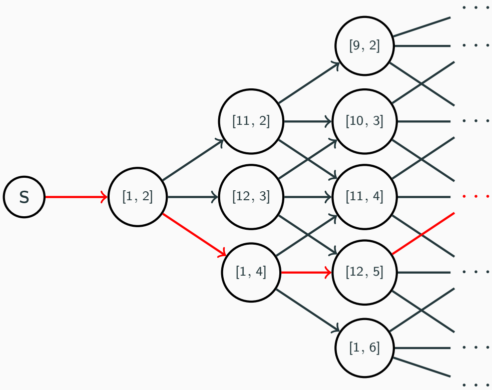

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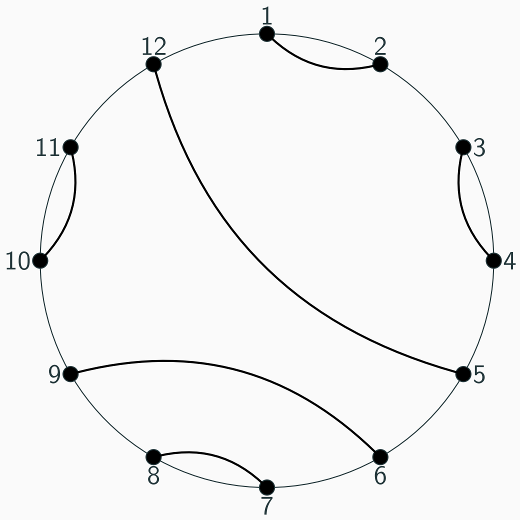
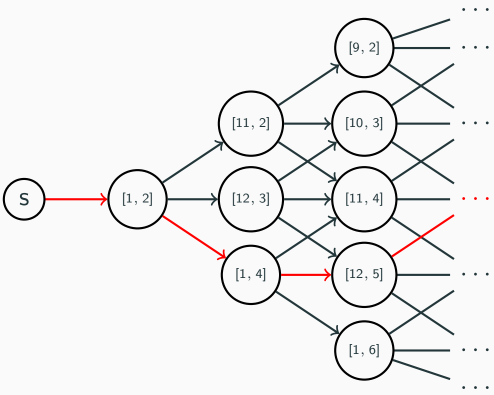


Every path corresponds to an element in $\mathbf{P}_{d/2}$.



The Hard Polynomial

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The new pair: $(3, 4)$.

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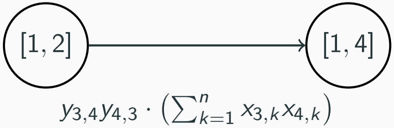

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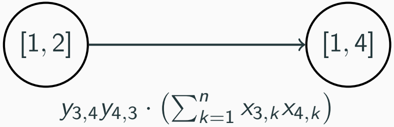

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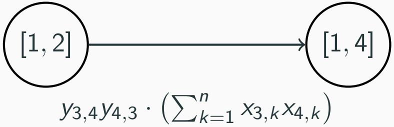

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$\left(\sum_{k=1}^{n} x_{3,k}x_{4,k}\right)$: To achieve full-rank.

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$y_{3,4}y_{4,3} \cdot \left(\sum_{k=1}^{n} x_{3,k}x_{4,k}\right)$

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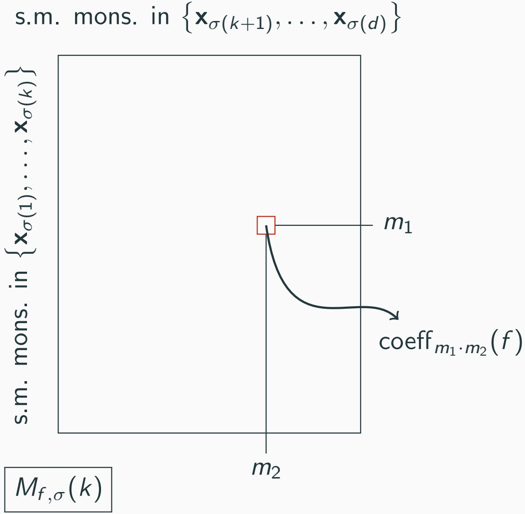
	$x_{4,1}$	$x_{4,2}$	$\cdots$	$\cdots$	$x_{4,n}$
$x_{3,1}$	1	0	$\cdots$	$\cdots$	0
$x_{3,2}$	0	1	$\cdots$	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$			$\vdots$
$\vdots$	$\vdots$	$\vdots$			$\vdots$
$x_{3,n}$	0	0	$\cdots$	$\cdots$	1

Lower Bound for a single osmABP



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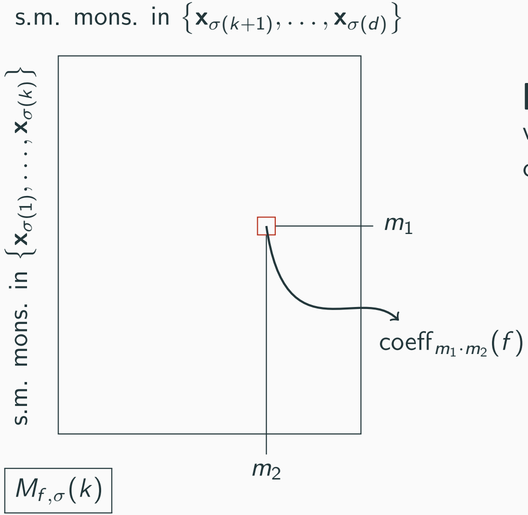

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[Nisan 91]: For every $1 \leq k \leq d$, the number of vertices in the k-th layer of the smallest osmABP(σ) computing f is equal to the rank of $M_{f,\sigma}(k)$.

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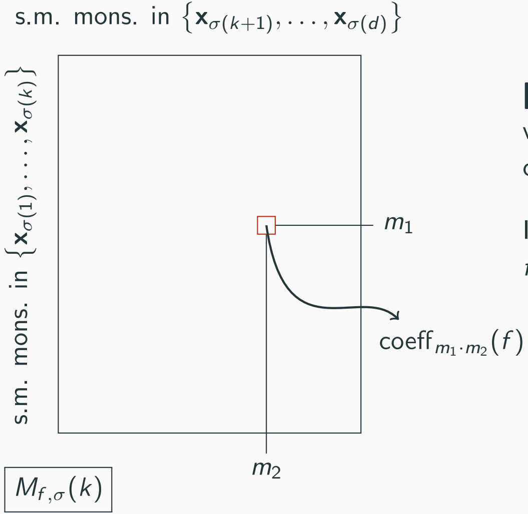

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If $\mathcal{A}$ is the smallest osmABP (in order σ) computing f, then

$$\mathrm{size}(\mathcal{A}) = \sum_{i=1}^{d} \mathrm{rank}(M_{f,\sigma}(k)).$$

Lower Bound for a single osmABP (contd.)

$$G_{n,d} = \sum_{\mathcal{P} \in \mathbf{P}_{d/2}} \prod_{(i,j) \in \mathcal{P}} y_{i,j} y_{j,i} \cdot \left(\sum_{k=1}^{n} x_{i,k} x_{j,k} \right).$$

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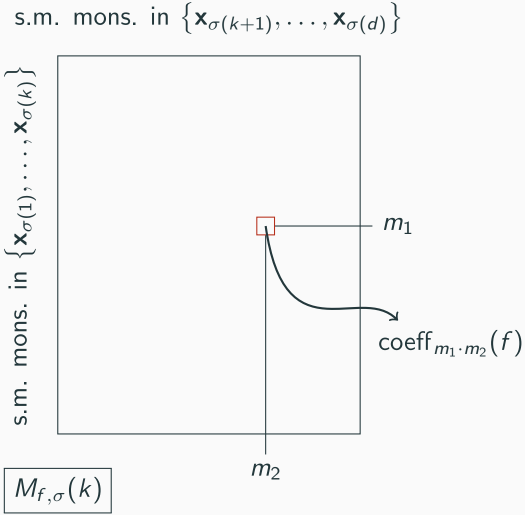

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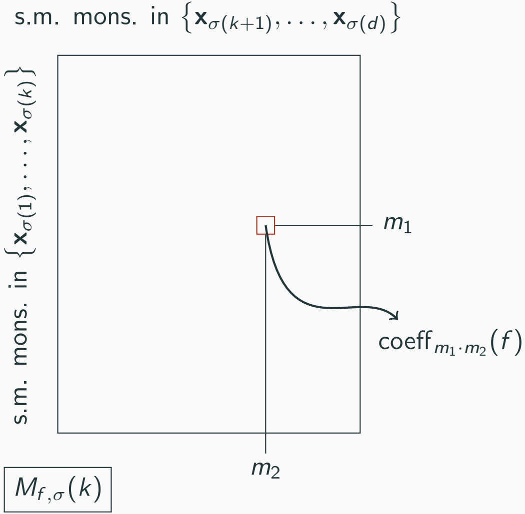
Lower Bound for a single osmABP (contd.)



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  Therefore,

$$\text{rank}(M_{G_{n,d},\sigma}(d/2)) = \Omega(n^{d/8}).$$

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Thank You!