Lower Bounds for some Algebraic Models of Computation

Prerona Chatterjee (Visiting Faculty Member, NISER Bhubaneswar)

November 11, 2024

Complexity Theory

Addition v.s. Multiplication v.s. Factorisation

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- study the amount of resource required by the model to complete the task.

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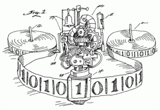
Given a boolean function f on n inputs, how many steps are required by a Turing machine to compute the f (in terms of n)?



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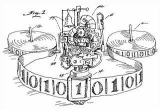

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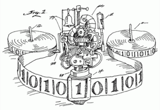

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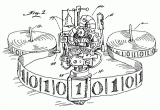

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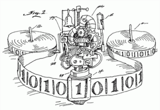

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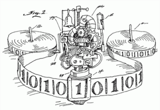

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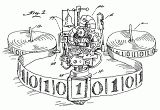

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[Shamir 79, Lipton 94]: If $h(x) = \prod_{i=1}^{d}(x - i)$ can be computed using $\text{poly}(\log d)$ additions and multiplications, then integer factoring is easy for boolean circuits.

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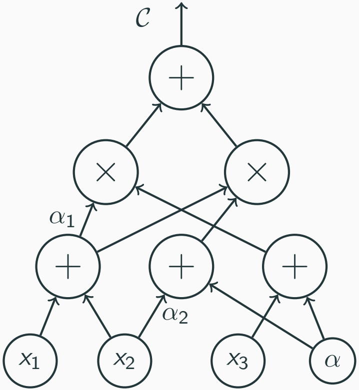

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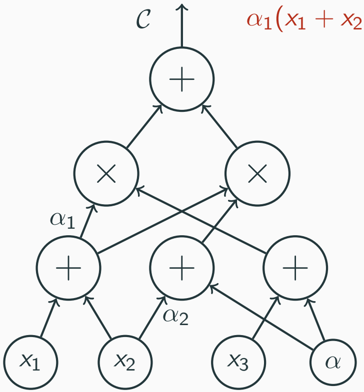

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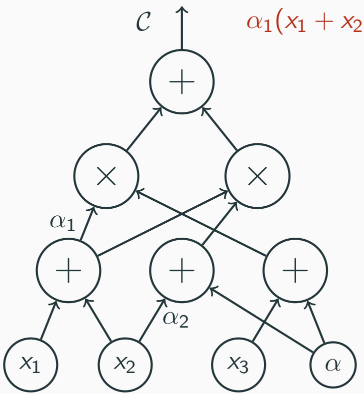

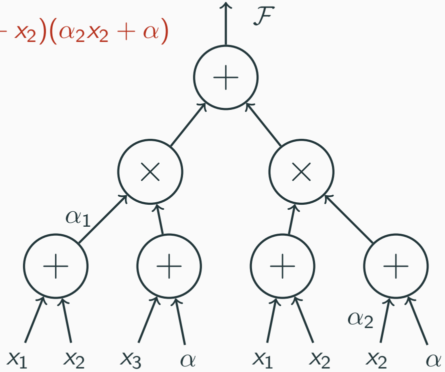

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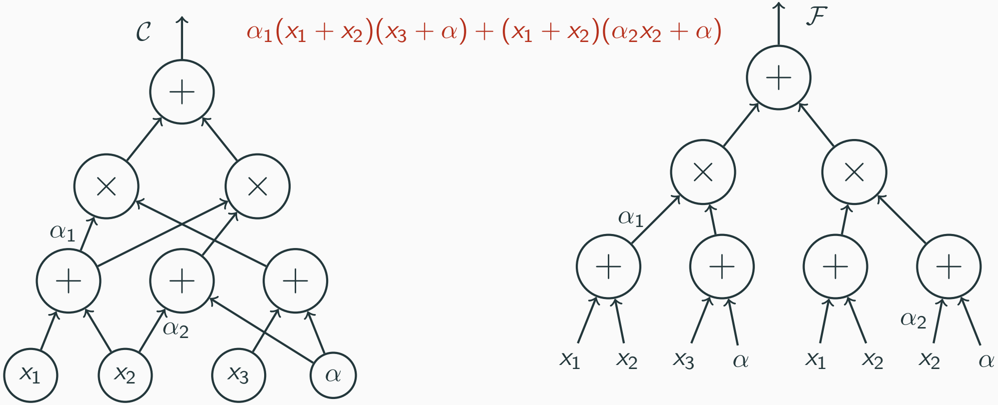
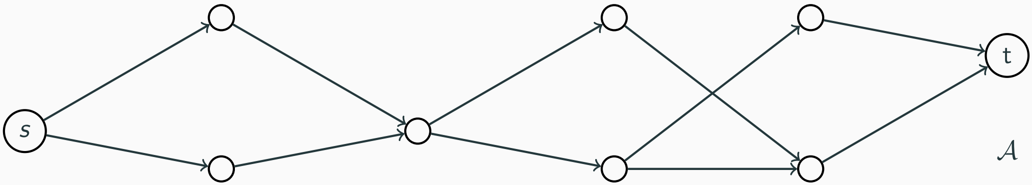


Q: Given $f(\mathbf{x}) \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d, how many $+, \times, -$ gates are needed to compute f?

Algebraic Branching Programs



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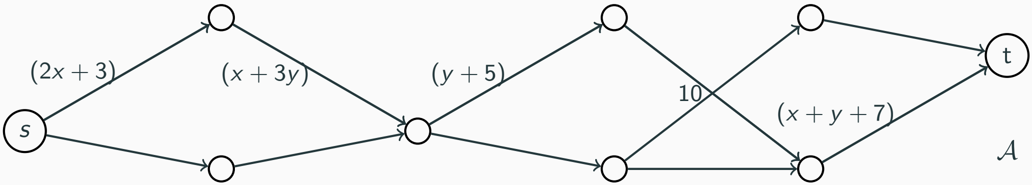

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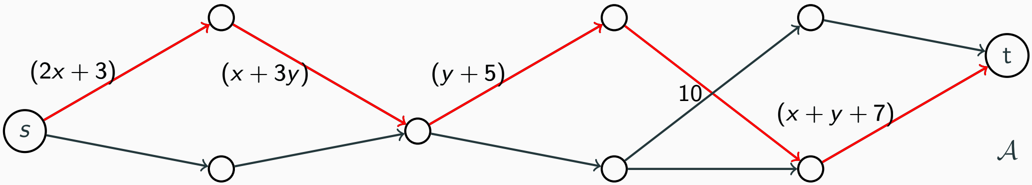

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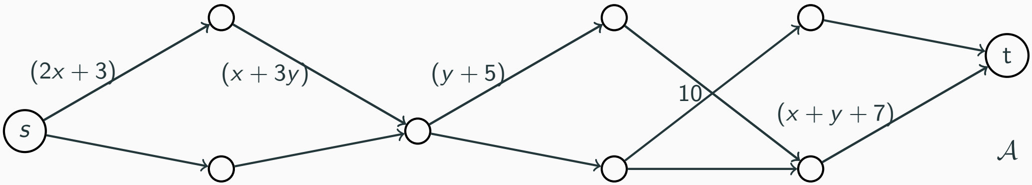

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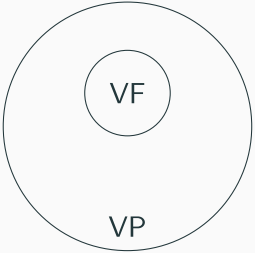

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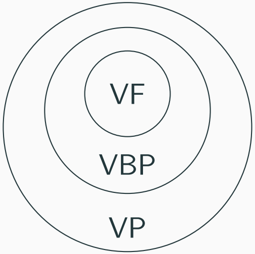

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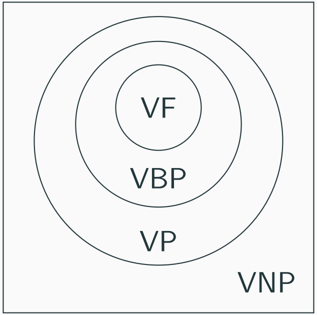

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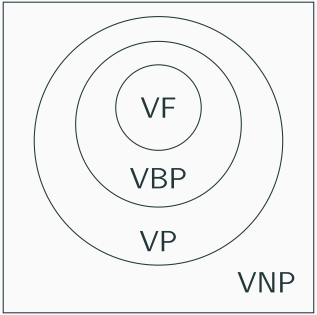

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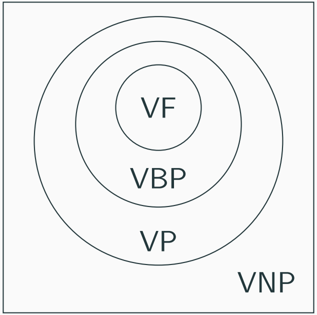

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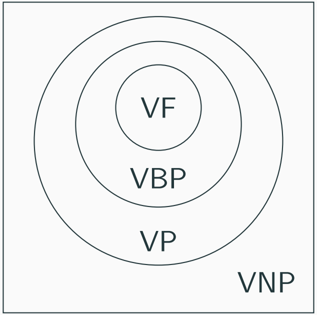

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Other Motivating Questions: Are the other inclusions tight?

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$$\mathrm{ESYM}_{n,d}(\mathbf{x}) = \sum_{i_1 < \cdots < i_d \in [n]} x_{i_1} \cdots x_{i_d}.$$

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[C-Kush-Saraf-Shpilka 24]: For $\omega(\log n) = d \leq n$, there is a polynomial $G_{n,d}(\mathbf{x})$ which is set-multilinear w.r.t $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$, where $|\mathbf{x}_i| \leq n$ for every $i \in [d]$, such that:

- $G_{n,d}$ is computable by a set-multilinear ABP of size $\text{poly}(n)$,
- any $\sum$osmABP computing $G_{n,d}$ must have super-polynomial total-width.

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Further, there is a non-commutative circuit of size $O(n \log^2 n)$ that computes $\mathrm{OSym}_{n,n/2}(\mathbf{x})$.

ABP vs Formula in the Non-Commutative Setting

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If an n-variate polynomial is abecedarian with respect to $\{X_1, \ldots, X_m\}$ for $m = \log n$, then any formula computing f can be made abecedarian with only $\mathrm{poly}(n)$ blow-up in size.

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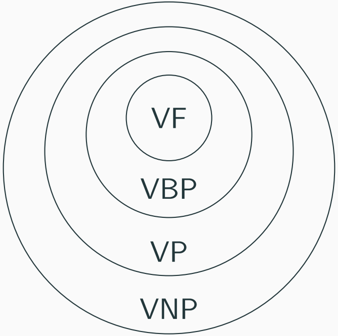

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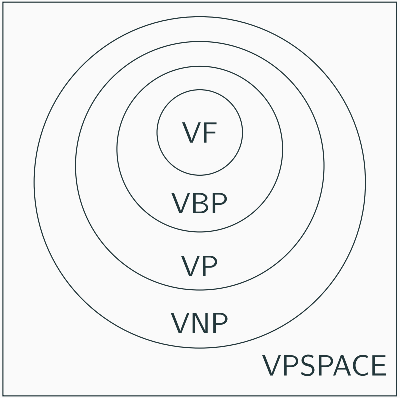

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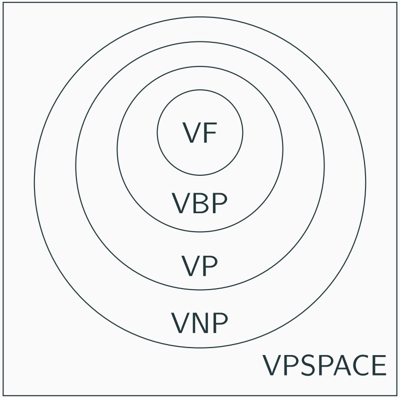

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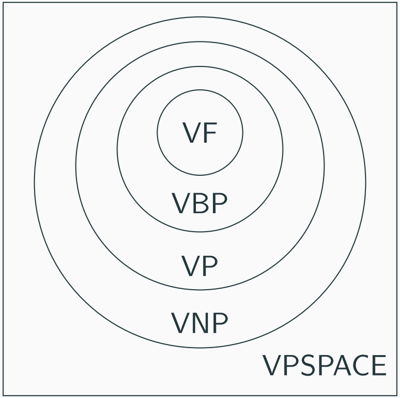

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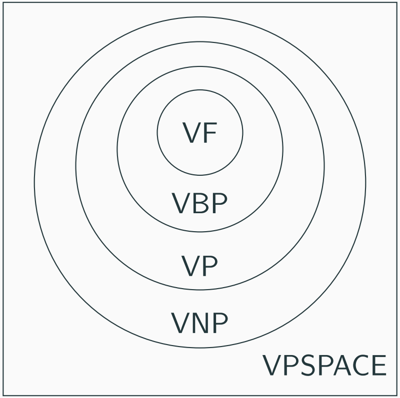
[C-Gajjar-Tengse 24]: $\mathsf{VNP} \neq \mathsf{VPSPACE}_b$ in the monotone setting.



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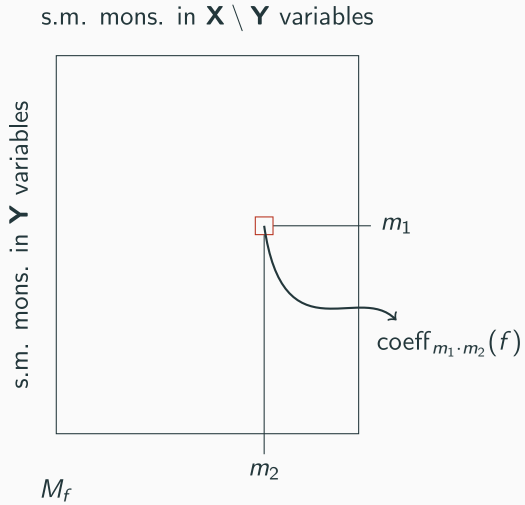

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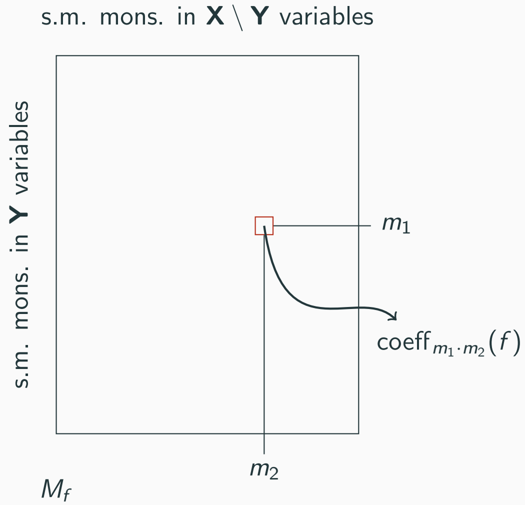

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The property "a matrix has low-rank" can be captured by a polynomial equation.

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Ongoing and Future Projects

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- Do VP have VP natural proofs under some reasonable conditions?

Branching Out

Study complexity theoretic questions about Boolean Circuits, Communication Models.

Teaching etc.

Courses I would be happy to teach

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- MA5320: Algebra I
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- MA5380: Topology
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Thank you!!!