Lower Bounds for some Algebraic Models of Computation

Prerona Chatterjee

April 12, 2024

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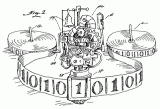

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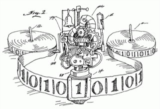

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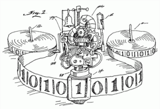

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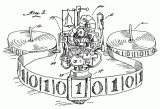

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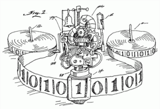

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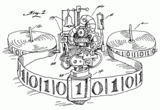

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Quantum Computation

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Matrix Multiplication Exponent (ω): Smallest number k such that the product of two $n \times n$ matrices can be found using n^k multiplications.

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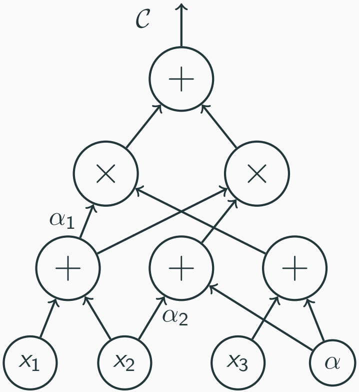

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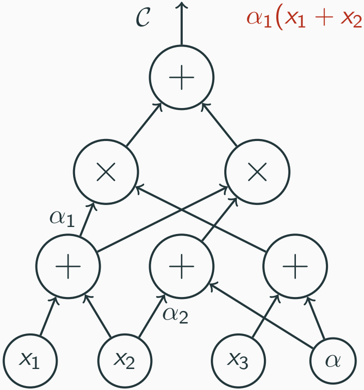

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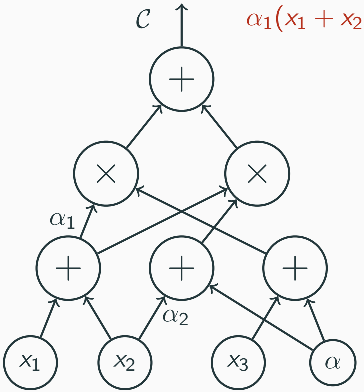

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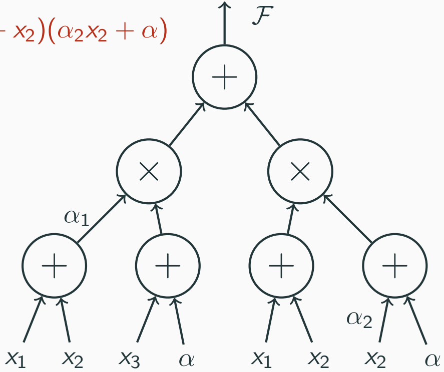

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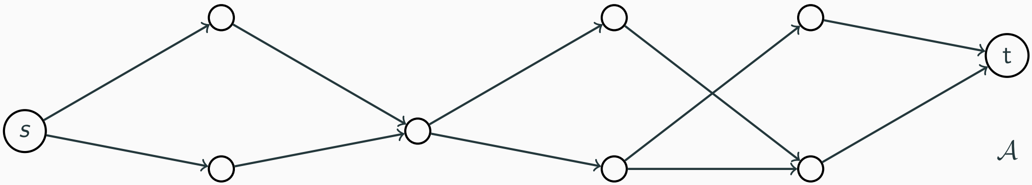

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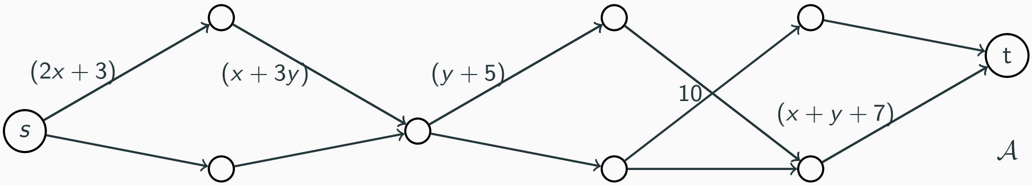

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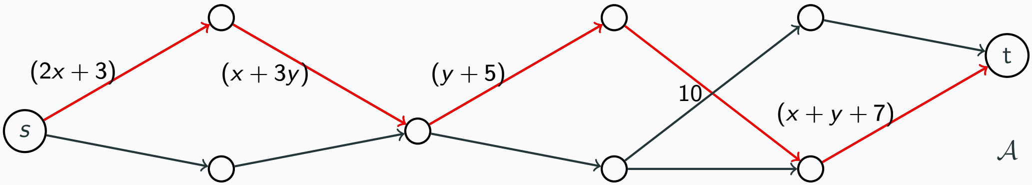

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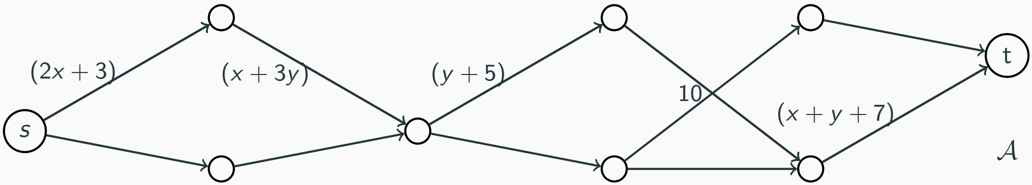

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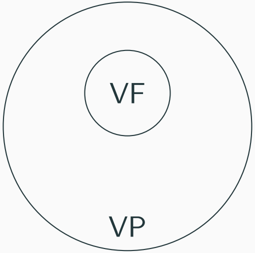

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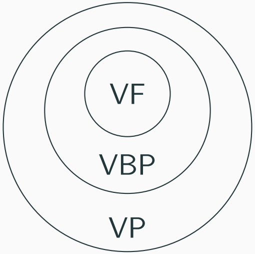

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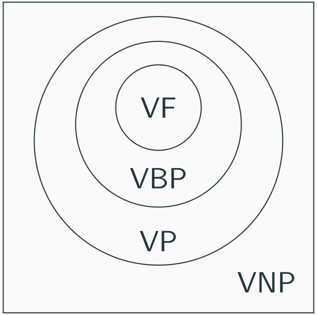

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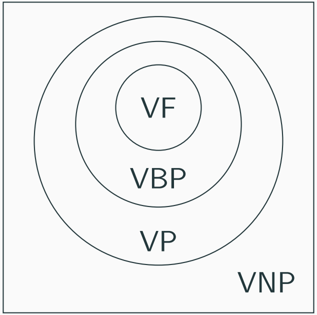

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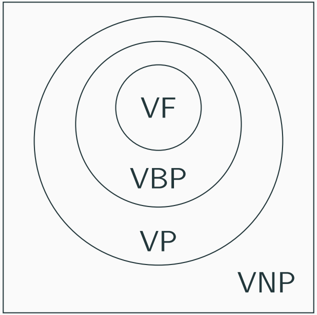

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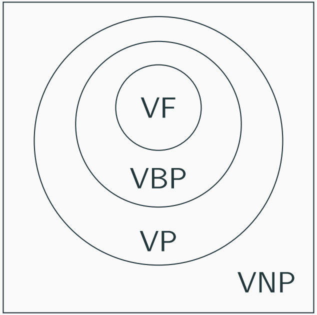

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Other Motivating Questions: Are the other inclusions tight?

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$$\mathrm{ESYM}_{n,d}(\mathbf{x}) = \sum_{i_1 < \cdots < i_d \in [n]} x_{i_1} \cdots x_{i_d}.$$

How does one make progress?

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Show that if a structured n-variate, degree-d polynomial is computable by a general model of size s, then they can also be computed by a structured model of size $\text{func}(s, n, d)$ for some function func.

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The lower bound is $n^{\Omega(\sqrt{d})}$ for depth-3 and depth-4.

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[C-Kush-Saraf-Shpilka 24]: For $\omega(\log n) = d \leq n$, there is a polynomial $G_{n,d}(\mathbf{x})$ which is set-multilinear w.r.t $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$, where $|\mathbf{x}_i| \leq n$ for every $i \in [d]$, such that:

- $G_{n,d}$ is computable by a set-multilinear ABP of size $\mathrm{poly}(n)$,
- any $\sum$osmABP computing $G_{n,d}$ must have super-polynomial total-width.

Set-Multilinearity

The variable set is divided into buckets.

$$\mathbf{x} = \mathbf{x}_1 \cup \cdots \cup \mathbf{x}_d \quad \text{where} \quad \mathbf{x}_i = \left\{x_{i,1}, \ldots x_{i,n_i}\right\}.$$

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computes a set-multilinear monomial with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$.

Near Tightness of ABP Set-Multilinearisation

For $\sigma \in S_d$, an ABP is σ-ordered set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

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Near Tightness of ABP Set-Multilinearisation

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Further, there is a non-commutative circuit of size $O(n \log^2 n)$ that computes $\mathrm{OSym}_{n,n/2}(\mathbf{x})$.

ABP vs Formula in the Non-Commutative Setting

[Nisan 91]: Any ABP computing $\mathrm{Pal}_n(x_0, x_1) = \sum_{w \in \{0,1\}^{n/2}} \mathbf{x}_w \cdot \mathbf{x}_{w^R}$ has size $2^{\Omega(n)}$.

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position indices $\equiv$ bucket indices

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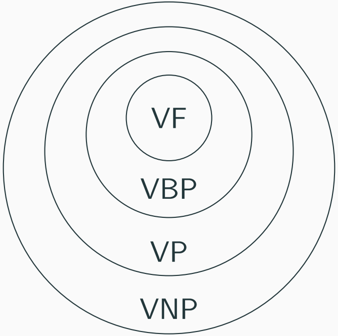
If an n-variate polynomial is abecedarian with respect to $\{X_1, \ldots, X_m\}$ for $m = \log n$, then any formula computing f can be made abecedarian with only $\text{poly}(n)$ blow-up in size.

Classes Beyond VNP



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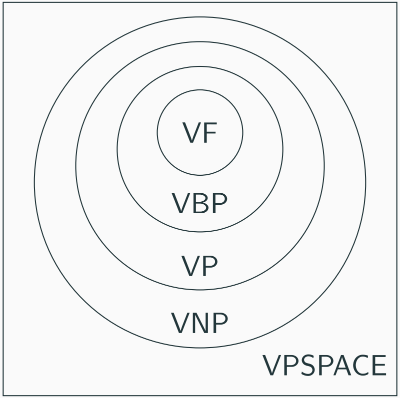
$\mathsf{VPSPACE}_b$: Polynomials whose coefficients can be computed in PSPACE/ poly and have degree bounded by poly(n).



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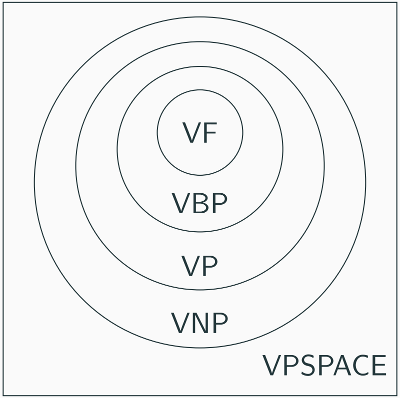

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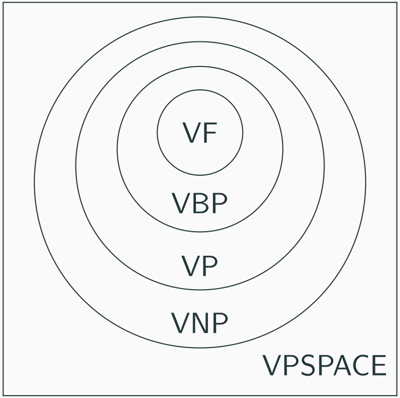

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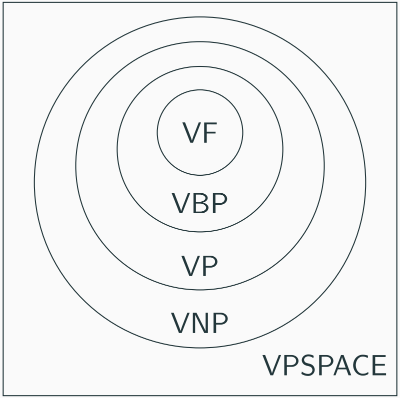
[C-Gajjar-Tengse 23]: $\mathsf{VNP} \neq \mathsf{VPSPACE}_b$ in the monotone setting.



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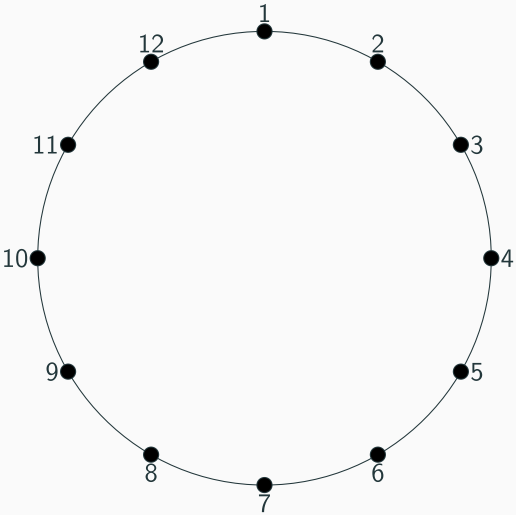
- any ordered set-multilinear branching program computing $G_{n,d}$ requires width $n^{\Omega(d)}$.

Arc Partition



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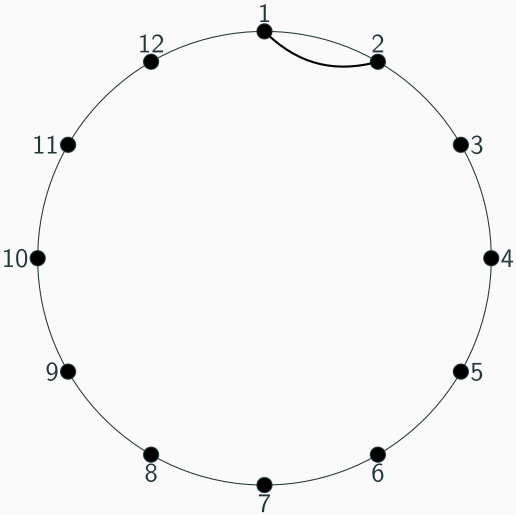

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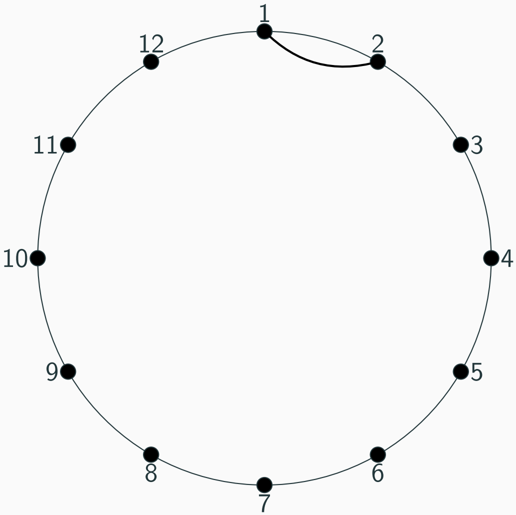

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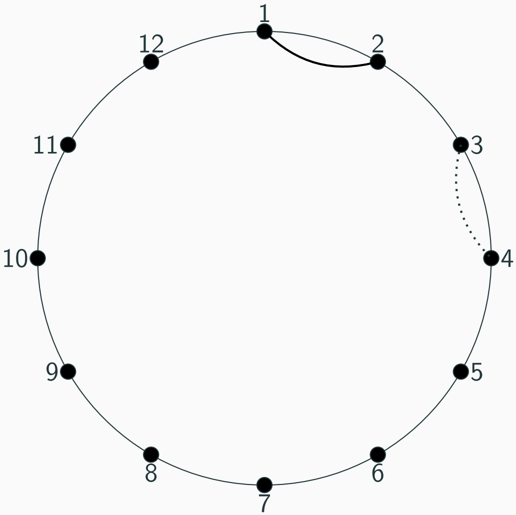

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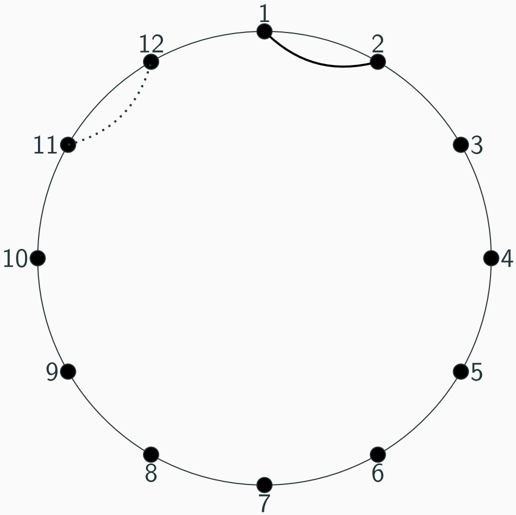

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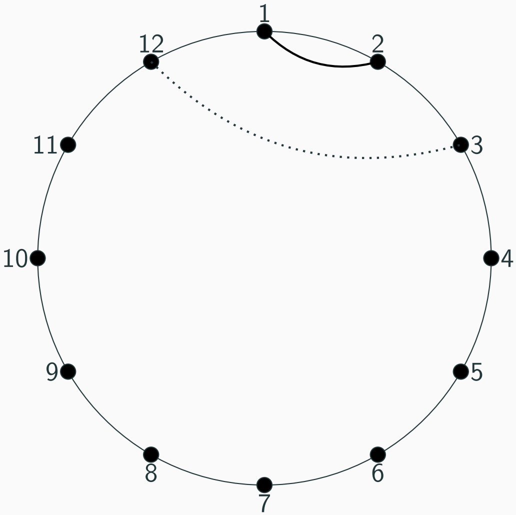

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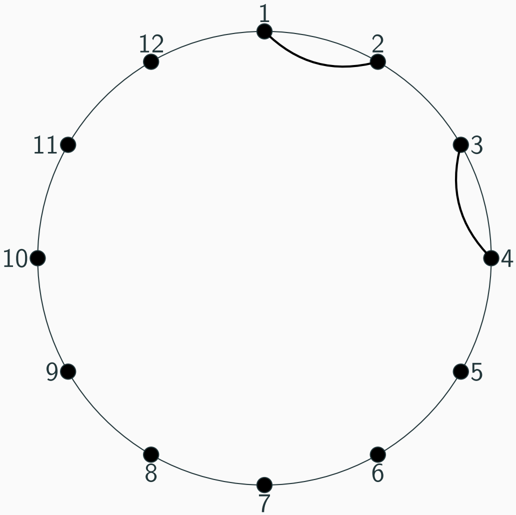

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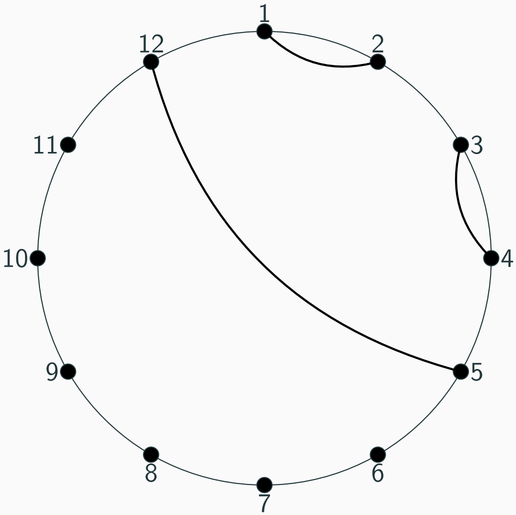

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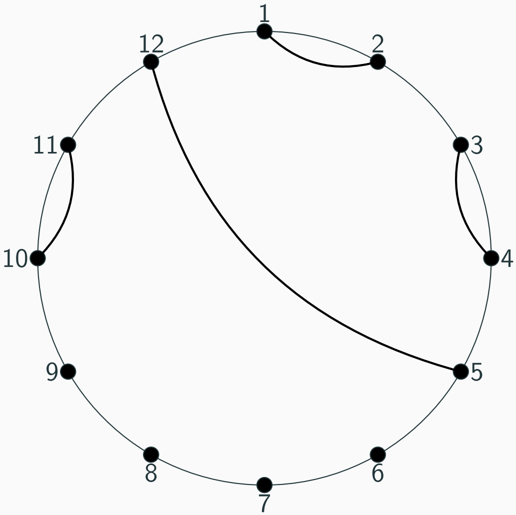

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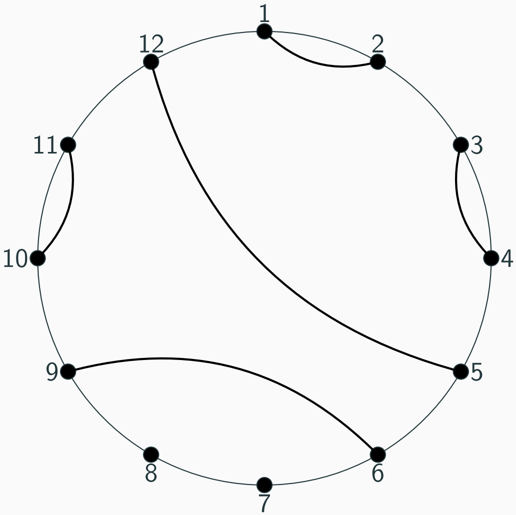

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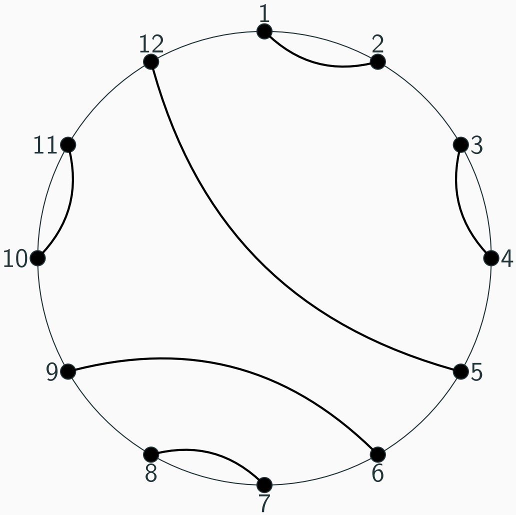
Arc Partition



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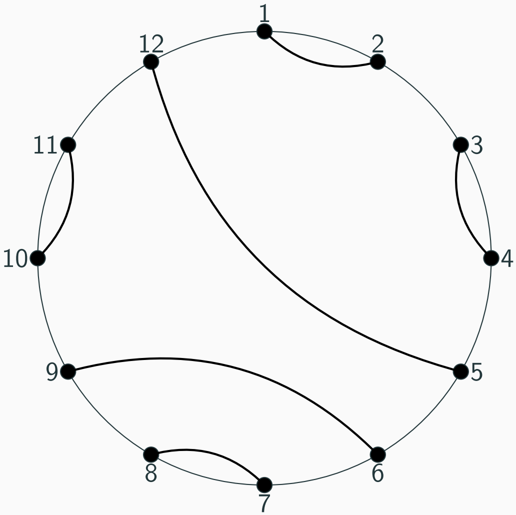

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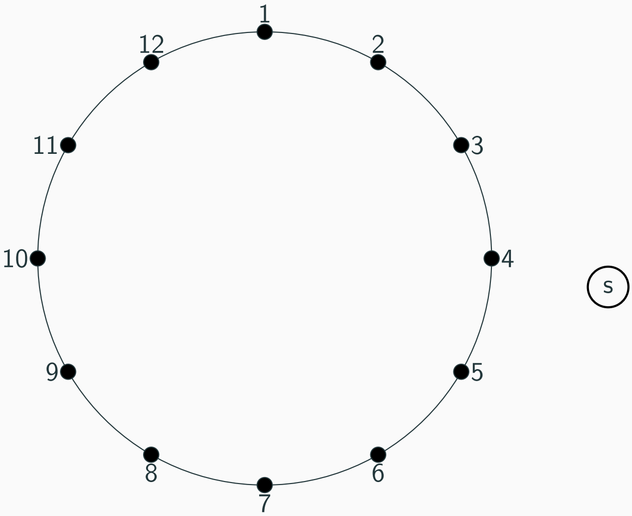
$\mathbf{P}_6 =$ All possibles sequences of such pairs.

The ABP Upper Bound



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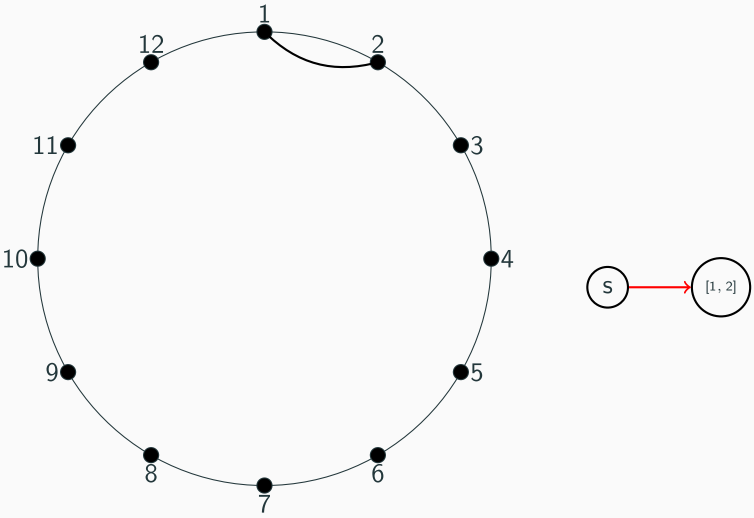

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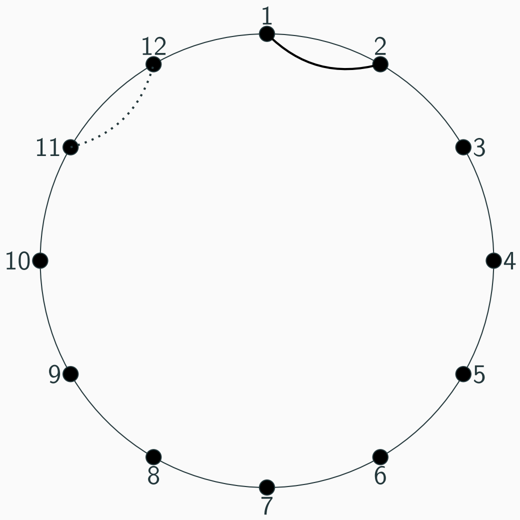

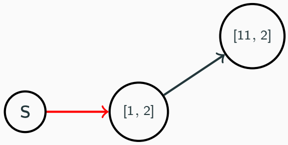

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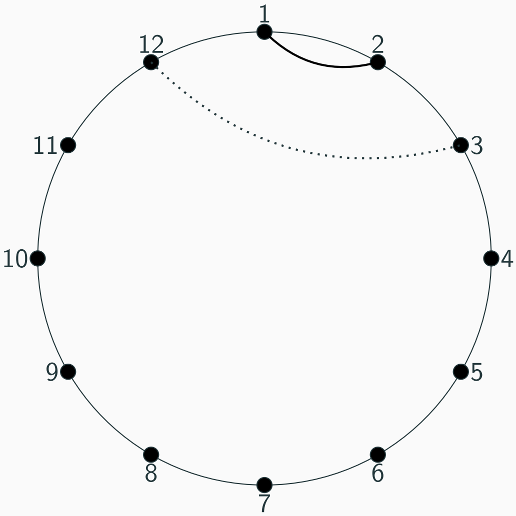

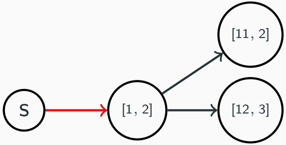

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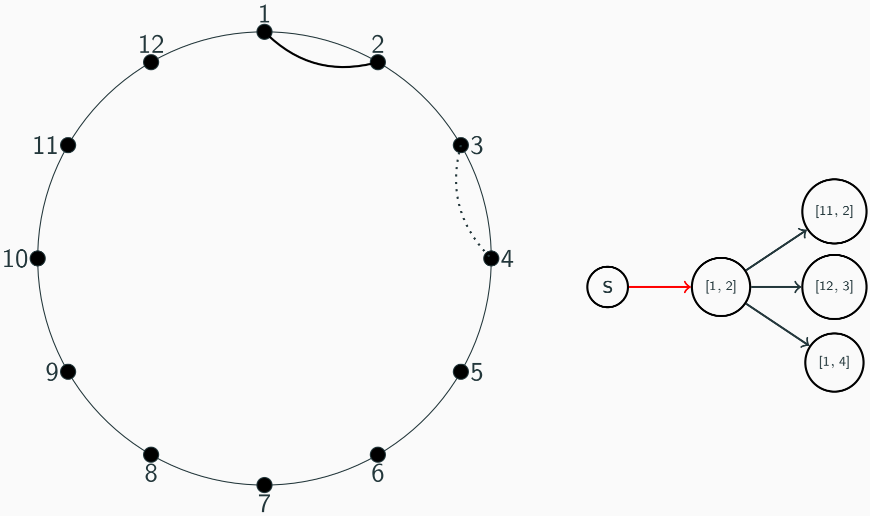

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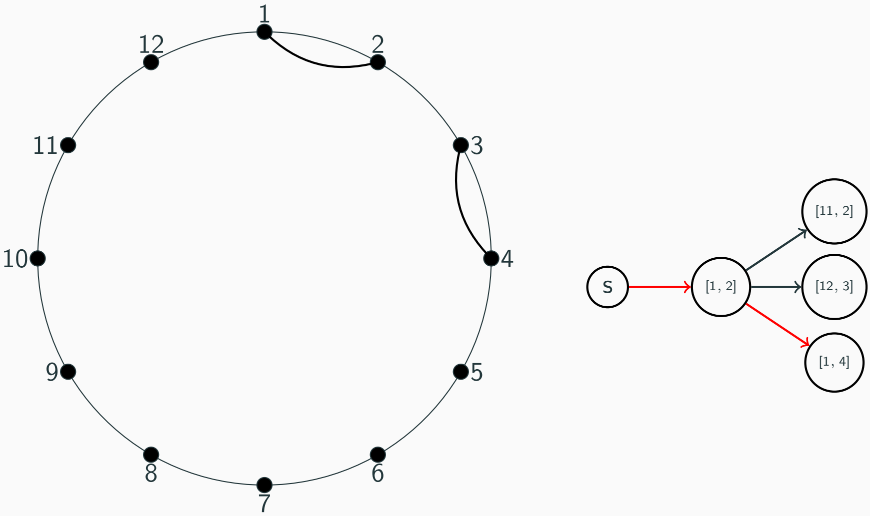

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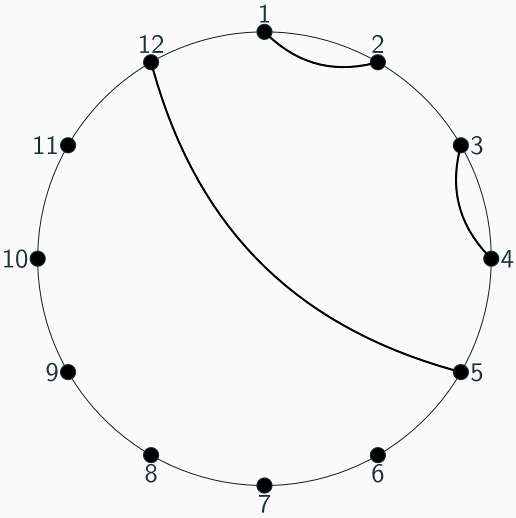

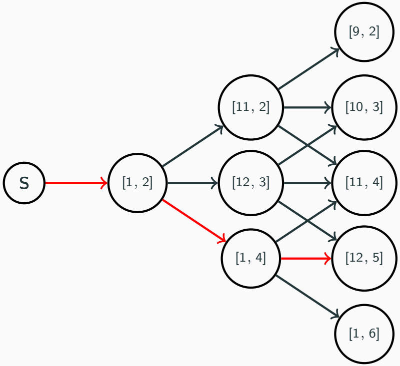

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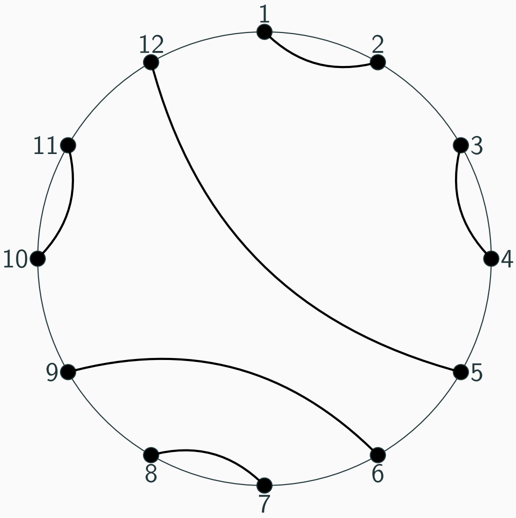

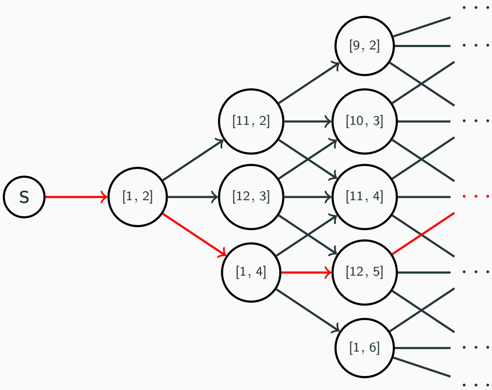

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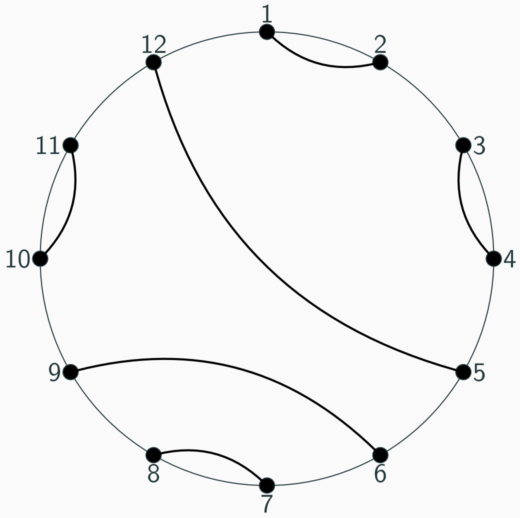
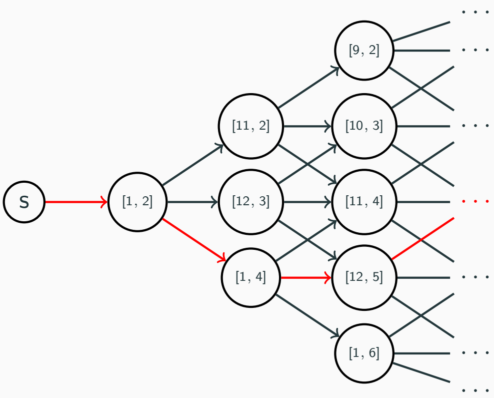


Every path corresponds to an element in $\mathbf{P}_{d/2}$.



The Hard Polynomial

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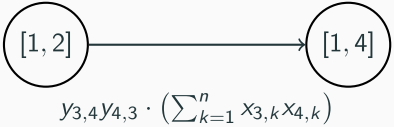
The new pair: $(3, 4)$.

The Hard Polynomial



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$y_{3,4}y_{4,3} \cdot \left(\sum_{k=1}^{n} x_{3,k}x_{4,k}\right)$

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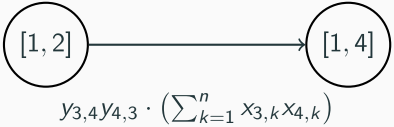

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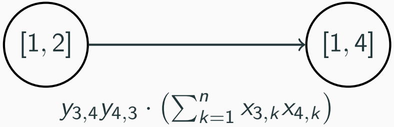

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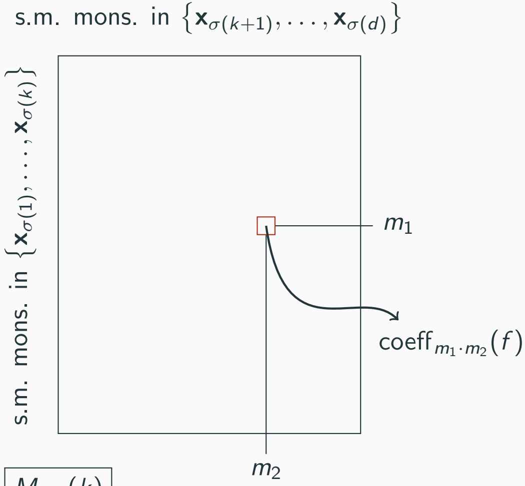
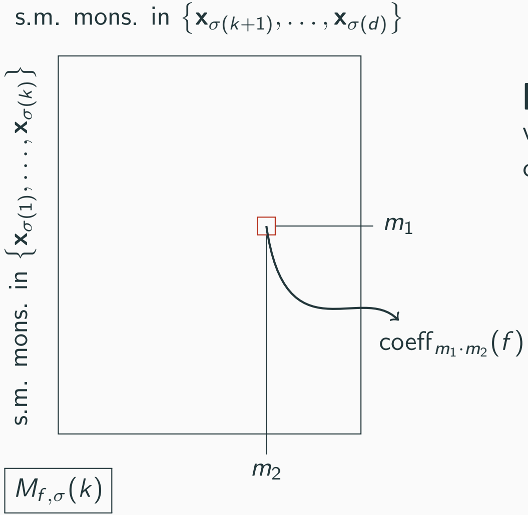
	$x_{4,1}$	$x_{4,2}$	$\cdots$	$\cdots$	$x_{4,n}$
$x_{3,1}$	1	0	$\cdots$	$\cdots$	0
$x_{3,2}$	0	1	$\cdots$	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$			$\vdots$
$\vdots$	$\vdots$	$\vdots$			$\vdots$
$x_{3,n}$	0	0	$\cdots$	$\cdots$	1

Lower Bound for a single osmABP

f is a set-multilinear poly. w.r.t $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$.



$M_{f,\sigma}(k)$



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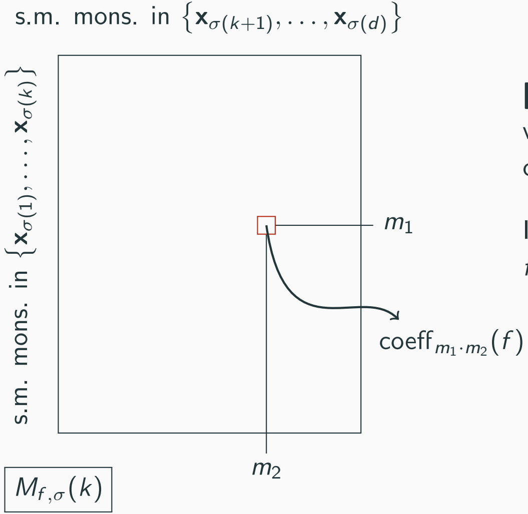

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If $\mathcal{A}$ is the smallest osmABP (in order σ) computing f, then

$$\text{size}(\mathcal{A}) = \sum_{i=1}^{d} \text{rank}(M_{f,\sigma}(k)).$$

Lower Bound for a single osmABP (contd.)

$$G_{n,d} = \sum_{\mathcal{P} \in \mathbf{P}_{d/2}} \prod_{(i,j) \in \mathcal{P}} y_{i,j} y_{j,i} \cdot \left(\sum_{k=1}^{n} x_{i,k} x_{j,k} \right).$$

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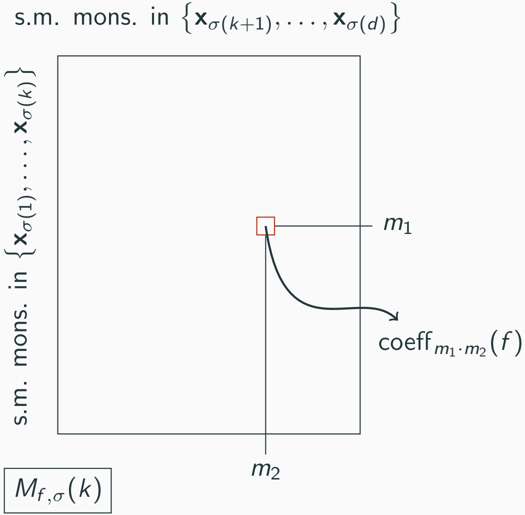

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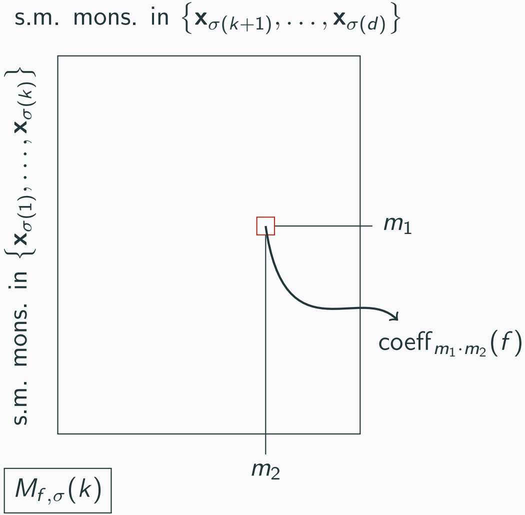

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  Therefore,

$$\text{rank}(M_{G_{n,d},\sigma}(d/2)) = \Omega(n^{d/8}).$$

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$\implies$ $M_w(G_{n,d})$ is far from full rank unless t is large.

Improved Lower Bound against Homogeneous Non-Commutative Circuits

$$f(x, y) = (x + y) \times (x + y) = x^2 + xy + yx + y^2 \neq x^2 + 2xy + y^2$$

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[Carmosino-Impagliazzo-Lovett-Mihajlin 18]

$\Omega(n^{\frac{\omega}{2}+\varepsilon})$ lower bound for an n-variate, degree-poly(n) polynomial $\implies$ arbitrarily large poly(n) lower bound for n-variate, degree-n polynomial.

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$$\mu(f) = \mathrm{rank}\left(\mathrm{span}_{\mathbb{F}}\left(\left\{f^{(0)}, f^{(1)}, \ldots, f^{(d)}\right\}\right)\right).$$

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Main Observation: If $f_1, \ldots, f_k$ are simultaneously computable by a homogeneous non-commutative circuit of size s,

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$\mathcal{C}$: Homogeneous non-commutative circuit.

$$\mu(\mathcal{C}) = \text{rank}\left(\text{span}_{\mathbb{F}}\left(\bigcup_{g \in \mathcal{C}} \left\{g^{(0)}, g^{(1)}, \ldots, g^{(d)}\right\}\right)\right).$$

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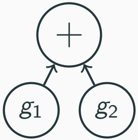

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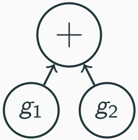

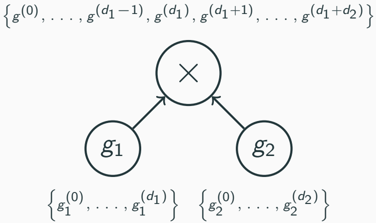

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