Recent Progress in Algebraic Circuit Complexity

Prerona Chatterjee

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Complexity of Computing Polynomials

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- **[Shamir 79, Lipton 94]**: If $h(x) = \prod_{i=1}^{d}(x - i)$ can be computed using $\mathrm{poly}(\log d)$ additions and multiplications, then integer factoring is easy for boolean circuits.

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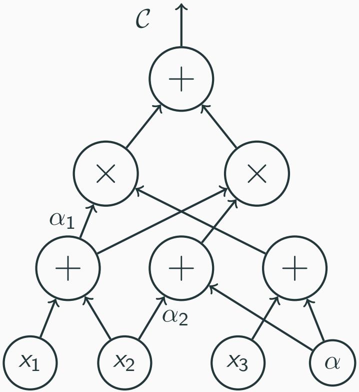

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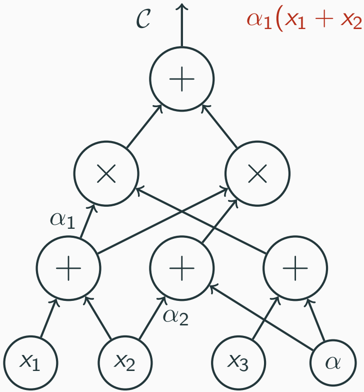

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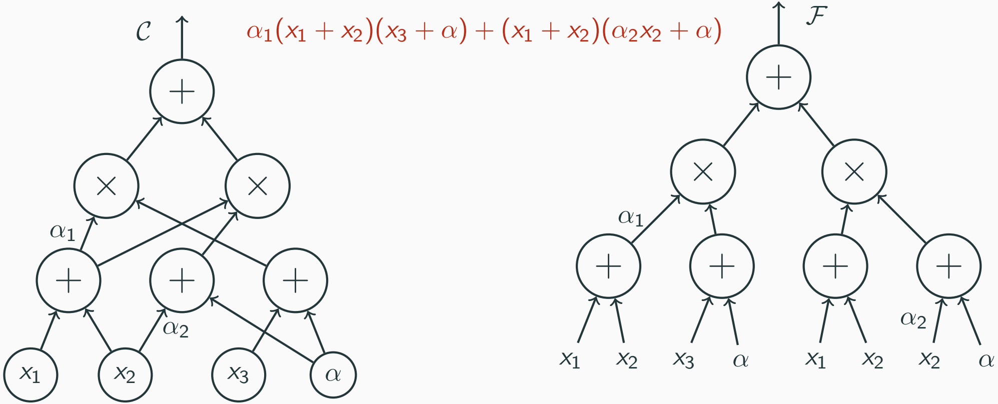

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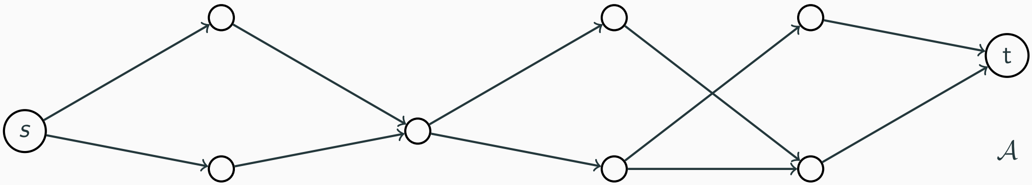

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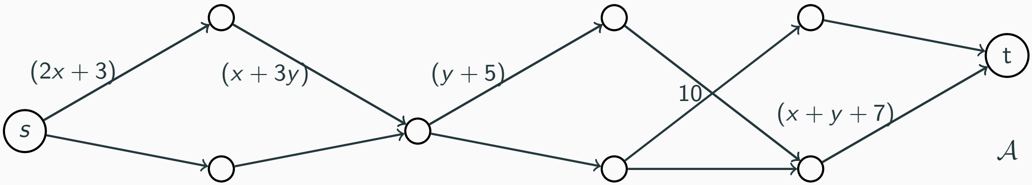

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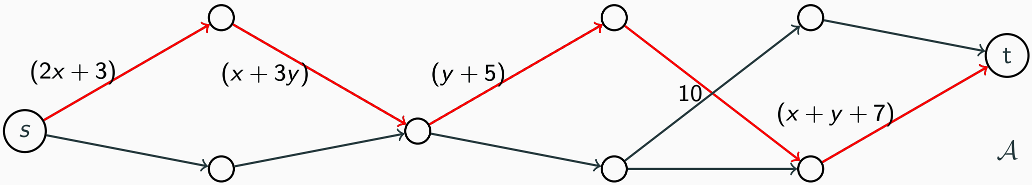

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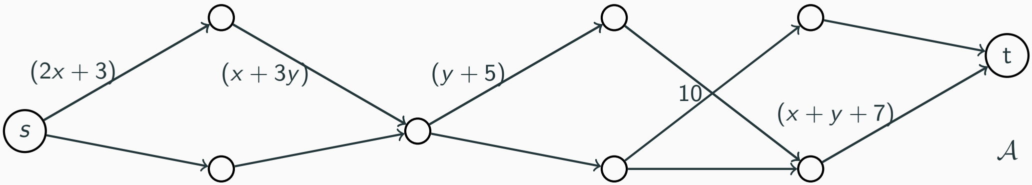

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- Polynomial computed by the ABP: $f_{\mathcal{A}}(\mathbf{x}) = \sum_p \mathrm{wt}(p)$

Determinant and Permanent Polynomials

Symbolic Matrix :

$$\begin{bmatrix} x_{11} & \cdots & x_{1j} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{i1} & \cdots & x_{ij} & \cdots & x_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nj} & \cdots & x_{nn} \end{bmatrix}$$

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Complete for the class of all explicit polynomials.

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$$\mathrm{ESYM}_{n,d} = \sum_{S\subseteq[n]} \prod_{i\in S} x_i = \mathrm{coeff}_{t^d}\left(\prod_{i\in[n]}(1+tx_i)\right)$$ is efficiently computable by $\Sigma\Pi\Sigma$ formulas.

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Iterated Matrix Multiplication

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Canonical example of a polynomial that is computable by an ABP of width n and length d.

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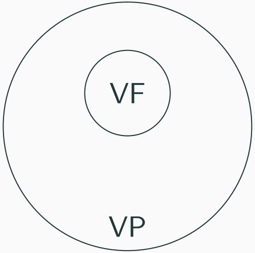

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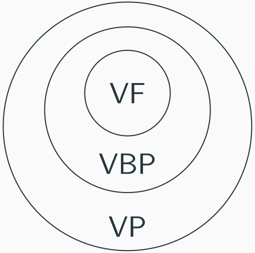

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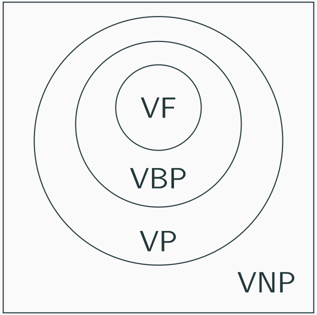

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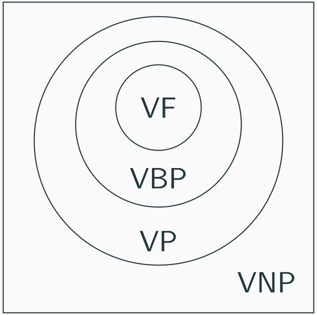

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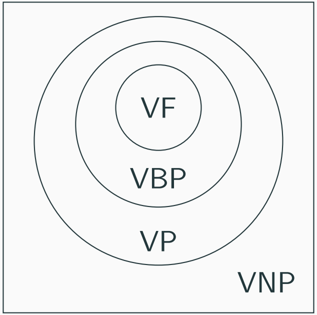

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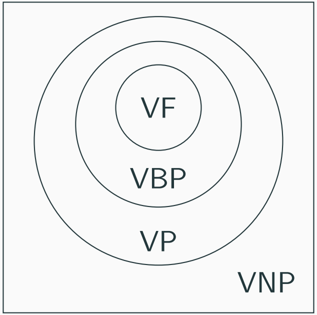

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Other Motivating Questions: Are the other inclusions tight?

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[C-Kumar-She-Volk 22]: Any formula computing $\mathrm{ESYM}_{n,0.1n}(\mathbf{x})$ requires $\Omega(n^2)$ vertices.

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Ultimate Goal: Prove better than $\text{func}(s, n, d)$ lower bounds.

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but is expected to require $n^{\Omega(\log d)}$-sized homogeneous formulas to compute.

[Fournier-Limaye-Srinivasan-Tavenas 24]
Any homogeneous non-commutative formula computing $\mathrm{ESYM}_{n,n/2}$ requires size $n^{\Omega(\log \log n)}$.

Set-Multilinearity

The variable set is divided into buckets.

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Circuits and ABPs: Can be set-multilinearised with $2^{O(d)}$ blow-up.

Homogenisation and Set-Multilinearisation of Formulas

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Self-Reducibility of $\mathrm{IMM}_{n,n}$

If $\mathrm{IMM}_{n,D}$ is computable by a set-multilinear formula of size s, then $\mathrm{IMM}_{n,d}$ is computable by a set-multilinear formula of size $s^{O(\frac{\log d}{\log D})}$.

Towards Proving General Formula Lower Bounds

Ways of proving Super-Polynomial Lower Bounds against Formulas:

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$n^{\Omega(\log n)}$ lower bound against set-multilinear formulas for $\mathrm{DMPY}_{n,n}$.

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[C-Kush-Saraf-Shpilka 24]

If $\mathrm{DMPY}_{n,n}$ were to be a set-multilinear projection of $\mathrm{IMM}_{w,n}$, then $w = n^{\Omega(n)}$.

Towards Proving General ABP Lower Bounds

[Bhargav-Dwivedi-Saxena 24]

Super polynomial lower bound against total-width of $\sum$osmABP for a polynomial of degree $d = O\left(\frac{\log n}{\log\log n}\right) \implies$ super-polynomial lower bound against ABPs.

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For $\omega(\log n) = d \leq n$, $\mathrm{DMPY}_{n,d}$ is a set-multilinear ABP of size $\mathrm{poly}(n)$, but any $\sum$osmABP computing $G_{n,d}$ must have super-polynomial total-width.

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In General (using the same techniques)

Size s circuits computing n-variate degree d polynomials can be converted into depth-Δ circuits of size $s^{O(d^{1/\Delta})}$.

[Limaye-Srinivasan-Tavenas 21]

Let d, n be such that $d \leq \frac{\log n}{100}$. For any $\Delta > 0$, any product-depth $\Delta \geq 1$ circuit computing $\mathrm{IMM}_{n,d}$ over any field of characteristic zero or $\geq d$, requires size $n^{\Omega\left(d^{\frac{1}{\exp(\Delta)}}\right)}$.

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[Forbes 24]: The theorem is true over all fields.

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Step 3: Prove a $n^{\Omega\left(\frac{d^{1/2^{\Delta}-1}}{\Delta}\right)}$ lower bound against set-multilinear constant depth circuits.

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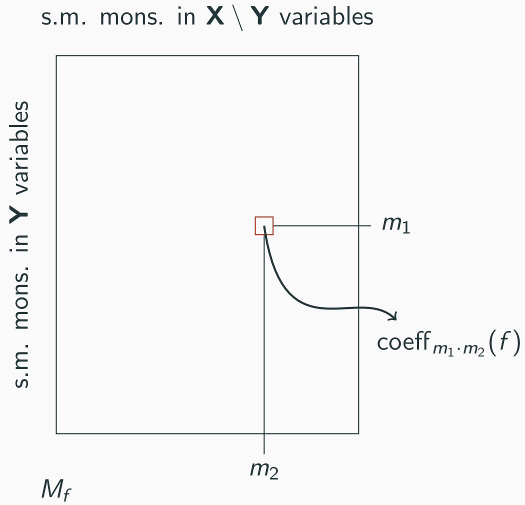

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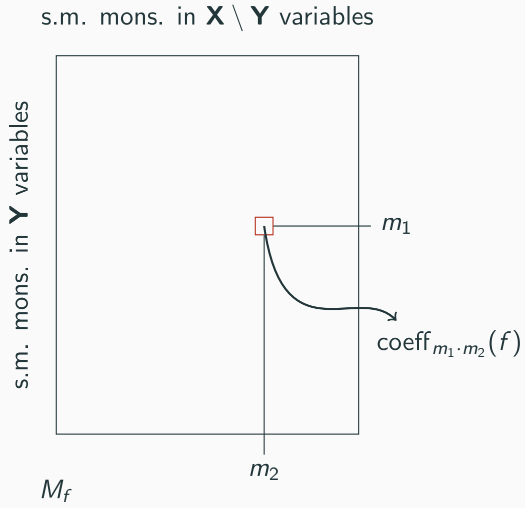

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The property "a matrix has low-rank" can be captured by a polynomial equation.

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The answer is not as clear as the boolean world.

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Next Big Goal of the Area: Proving lower bounds against Homogeneous Formulas.

Thank you!!!