Lower Bounds for some Algebraic Models of Computation

Prerona Chatterjee

December 2, 2024

Complexity Theory

Addition v.s. Multiplication v.s. Factorisation

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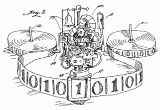
Given a boolean function f on n inputs, how many steps are required by a Turing machine to compute the f (in terms of n)?



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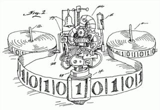

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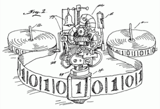

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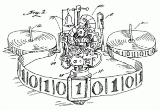

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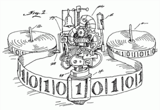

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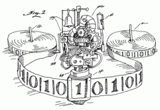

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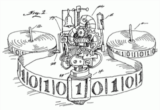

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[Shamir 79, Lipton 94]: If $h(x) = \prod_{i=1}^{d}(x - i)$ can be computed using $\text{poly}(\log d)$ additions and multiplications, then integer factoring is easy for boolean circuits.

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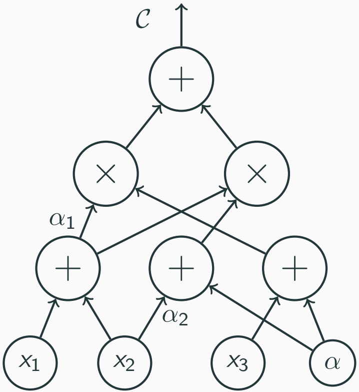

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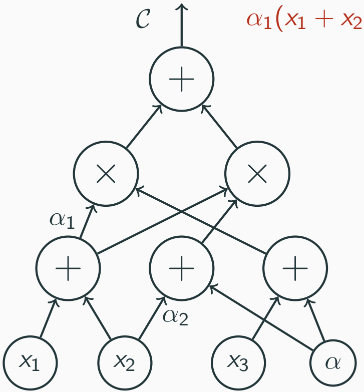

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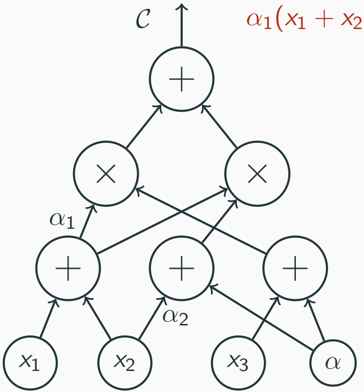

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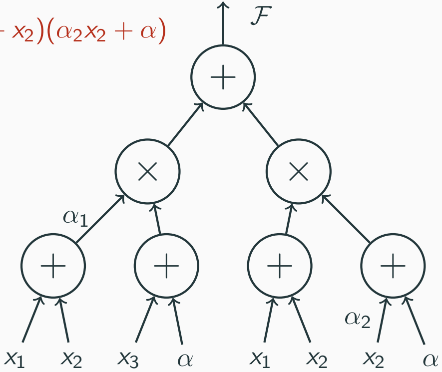

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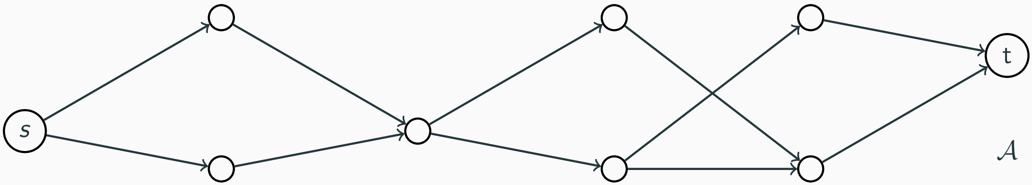

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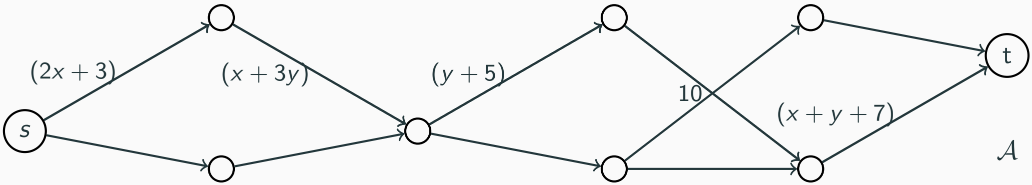

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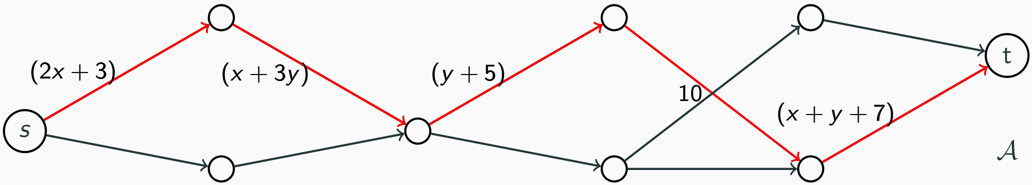

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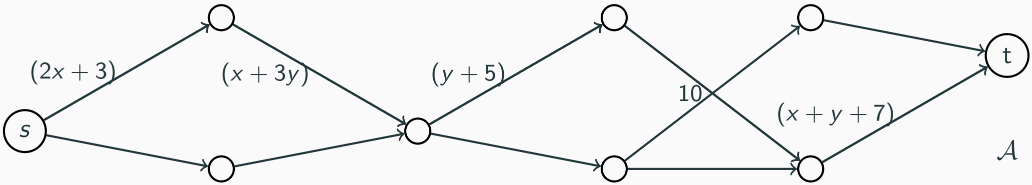

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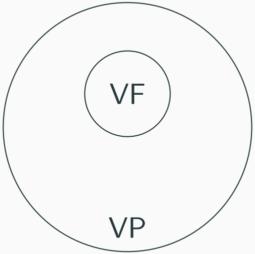

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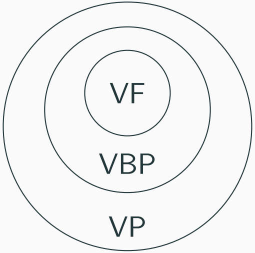

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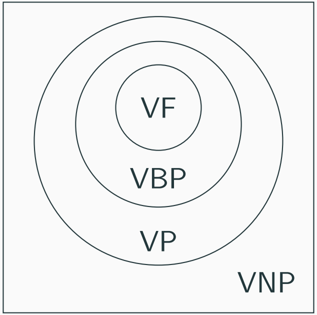

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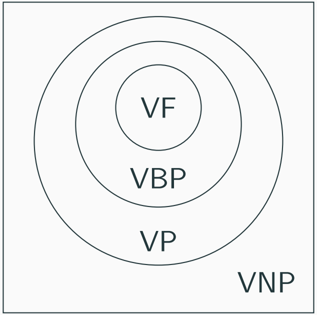

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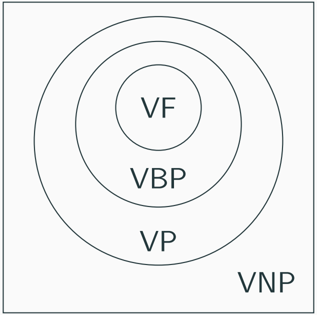

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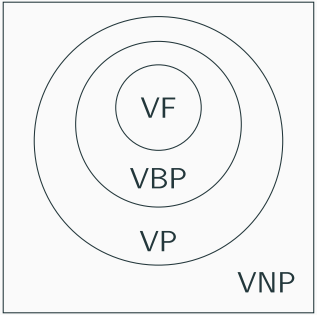

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Other Motivating Questions: Are the other inclusions tight?

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[C-Kumar-She-Volk 22]: Any formula computing $\mathrm{ESYM}_{n,0.1n}(\mathbf{x})$ requires $\Omega(n^2)$ vertices.

$$\mathrm{ESYM}_{n,d}(\mathbf{x}) = \sum_{i_1 < \cdots < i_d \in [n]} x_{i_1} \cdots x_{i_d}.$$

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Ultimate Goal: Prove better than $\mathrm{func}(s, n, d)$ lower bounds.

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[C-Kush-Saraf-Shpilka 24]: For $\omega(\log n) = d \leq n$, there is a polynomial $G_{n,d}(\mathbf{x})$ which is set-multilinear w.r.t $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$, where $|\mathbf{x}_i| \leq n$ for every $i \in [d]$, such that:

- $G_{n,d}$ is computable by a set-multilinear ABP of size $\mathrm{poly}(n)$,
- any $\sum$ osmABP computing $G_{n,d}$ must have super-polynomial total-width.

Set-Multilinearity

The variable set is divided into buckets.

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computes a set-multilinear monomial with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$.

Near Tightness of ABP Set-Multilinearisation

For $\sigma \in S_d$, an ABP is σ-ordered set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

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[C-K-S-S 24]: Super polynomial lower bound against total-width of $\sum$osmABP for a polynomial of degree $d = \omega(\log n)$ that is computable by polynomial-sized ABPs.

Non-Commutativity

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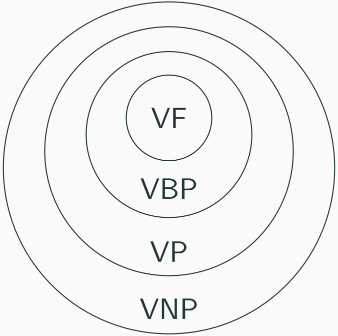
Classes Beyond VNP



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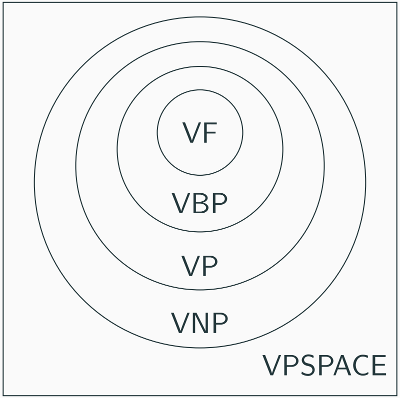

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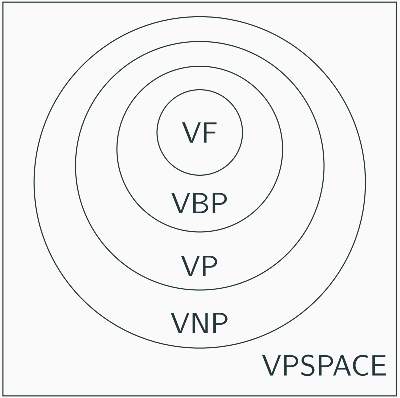

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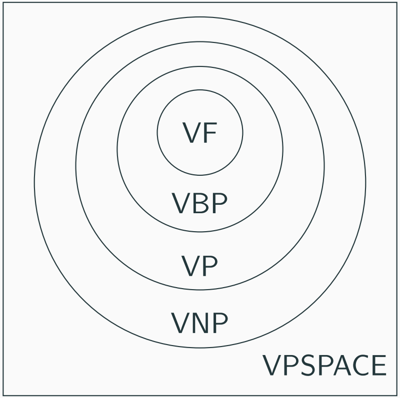

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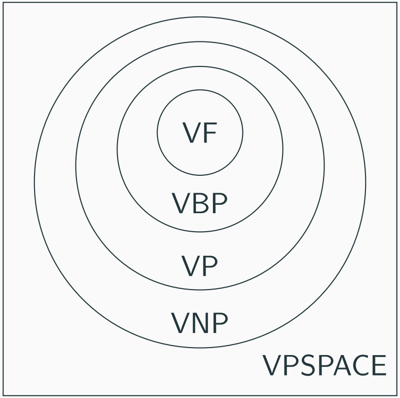
[C-Gajjar-Tengse 23]: $\mathrm{VNP} \neq \mathrm{VPSPACE}_b$ in the monotone setting.



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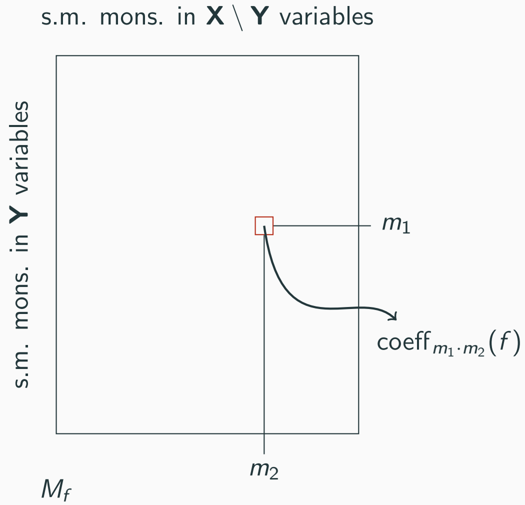

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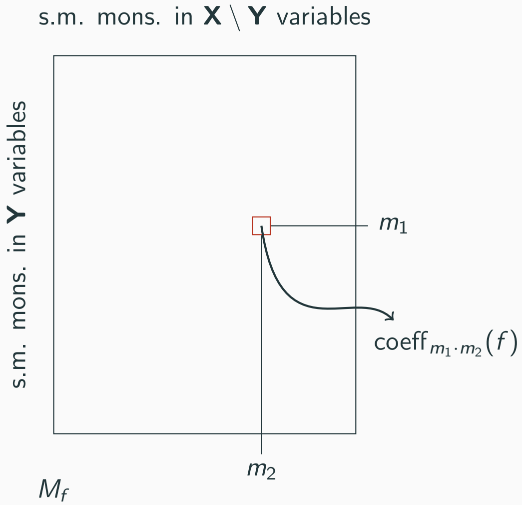

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The property "a matrix has low-rank" can be captured by a polynomial equation.

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Proof Overview of Lower Bound against Sum of osmABPs

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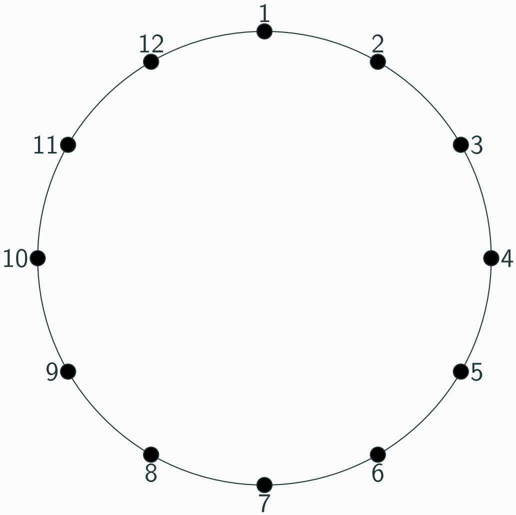
- any ordered set-multilinear branching program computing $G_{n,d}$ requires width $n^{\Omega(d)}$.

Arc Partition



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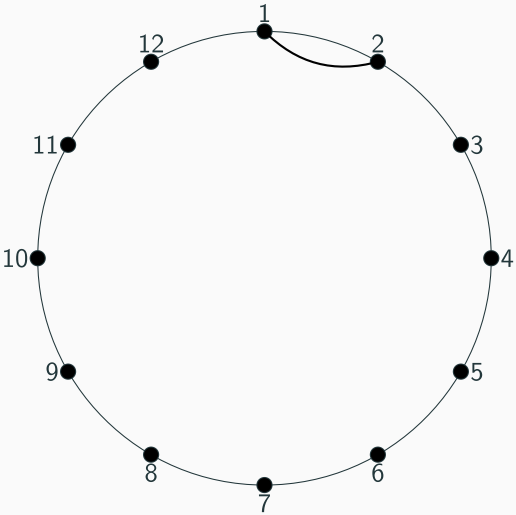

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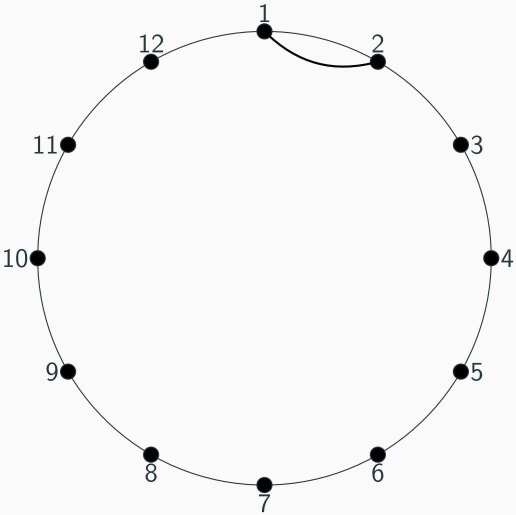
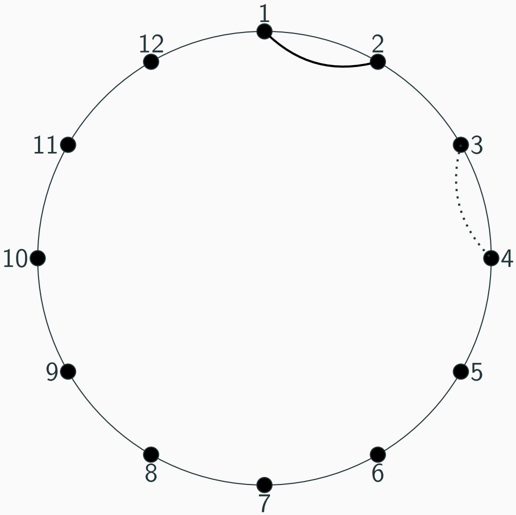


$\mathcal{P}_1 = \{(1, 2)\}$

Arc Partition



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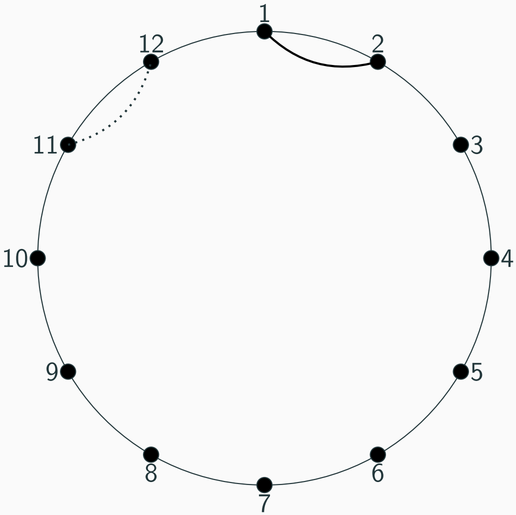

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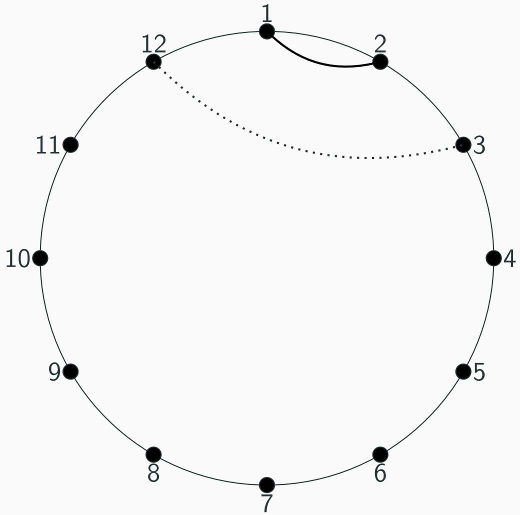

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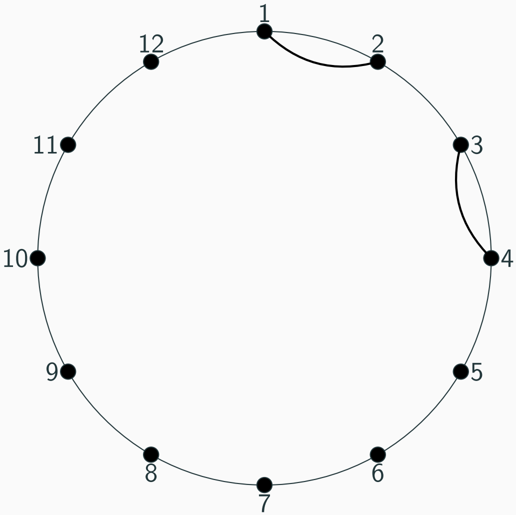
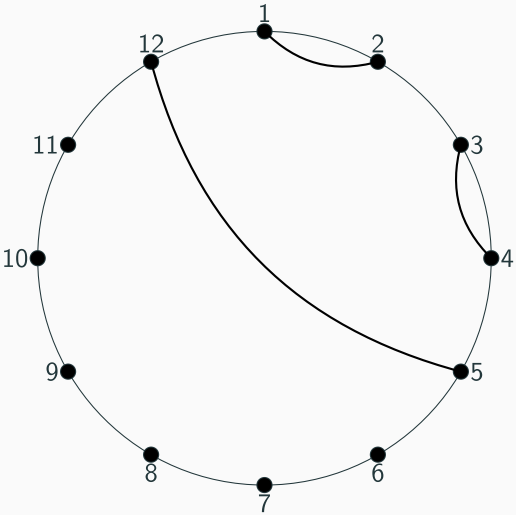


$$\mathcal{P}_1 = \{(1, 2)\}$$

$$\mathcal{P}_2 = \{(1, 2), (3, 4)\}$$

Arc Partition



$$\mathcal{P}_1 = \{(1,2)\}$$

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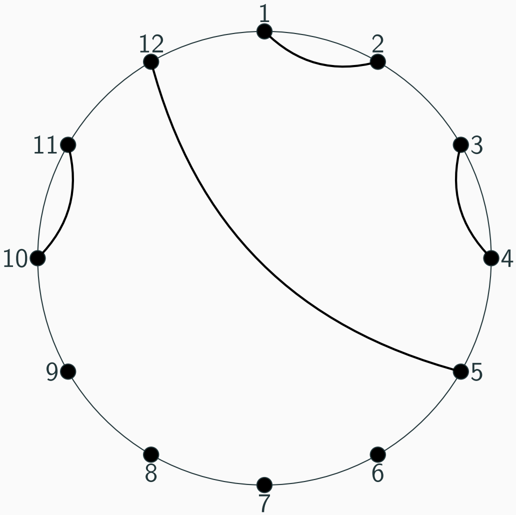
$$\mathcal{P}_3 = \{(1,2),(3,4),(12,5)\}$$

Arc Partition



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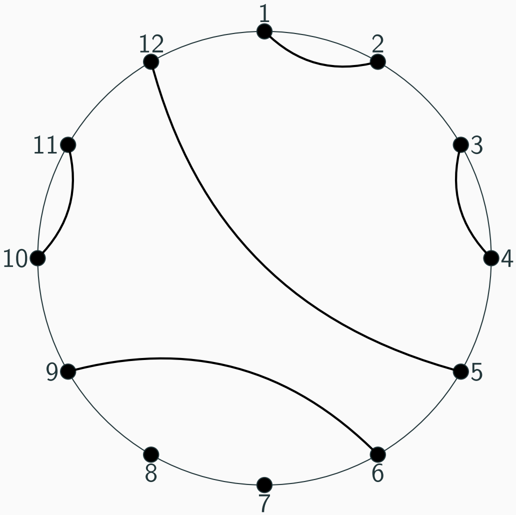
Arc Partition



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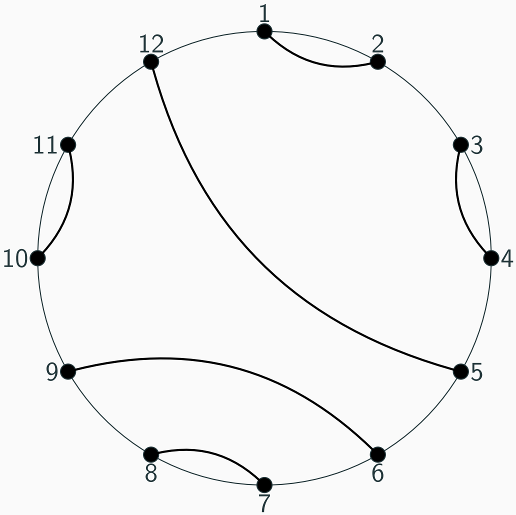
Arc Partition



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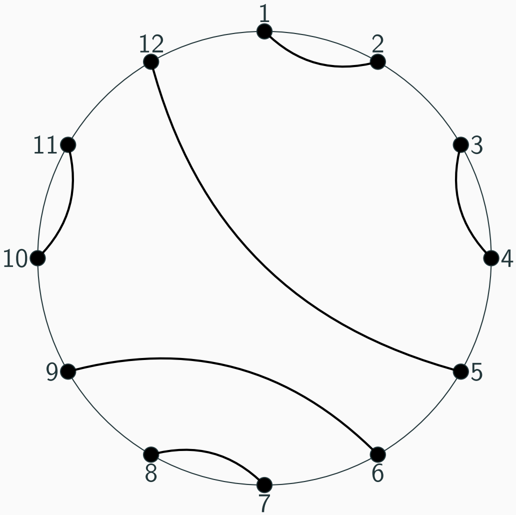
Arc Partition



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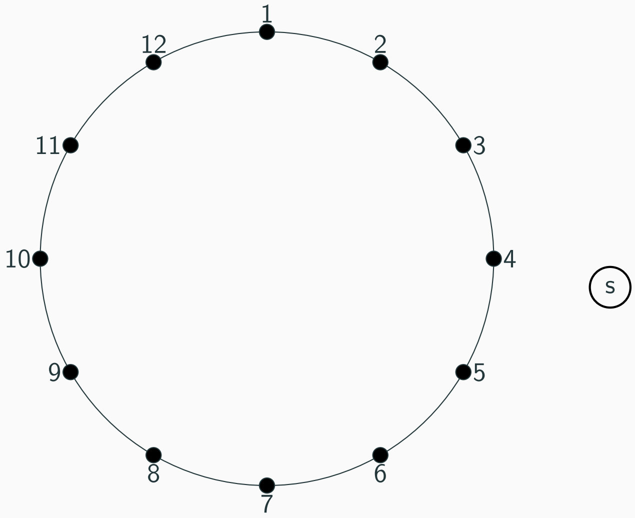
$\mathbf{P}_6 =$ All possibles sequences of such pairs.

The ABP Upper Bound



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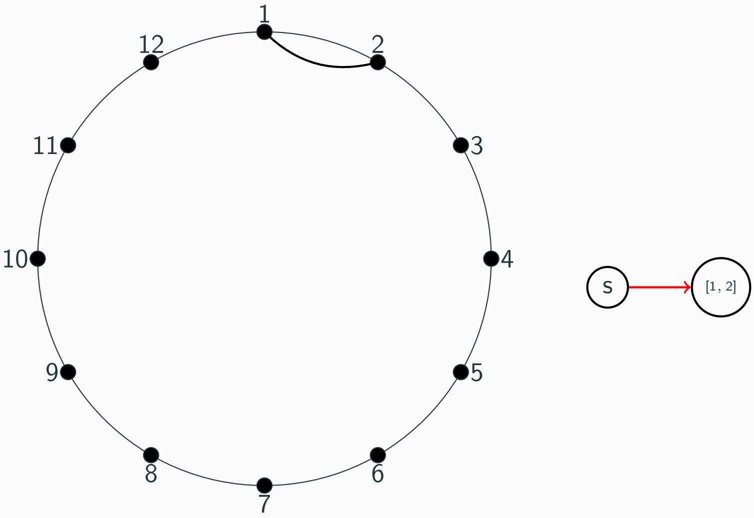

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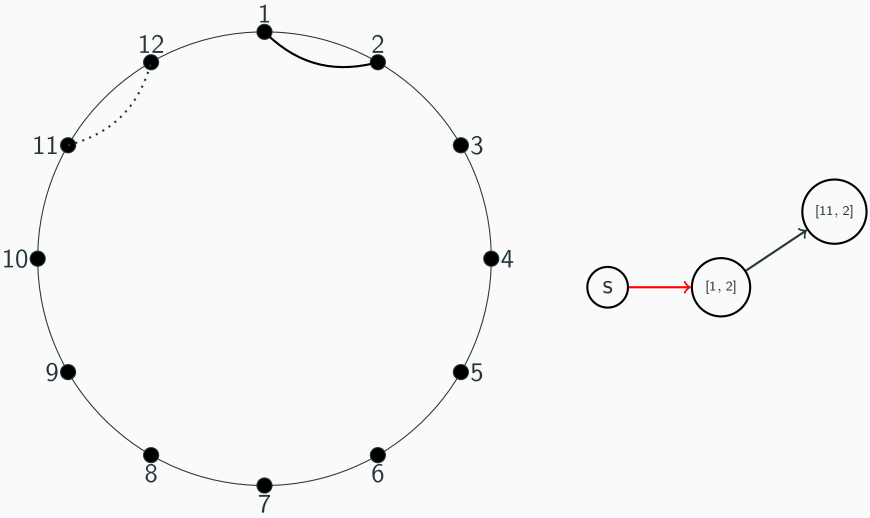

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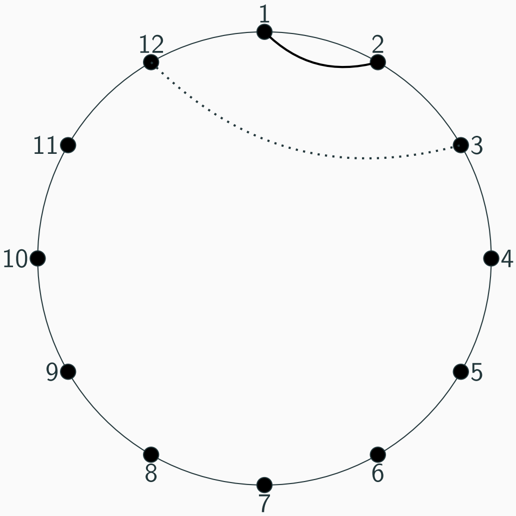

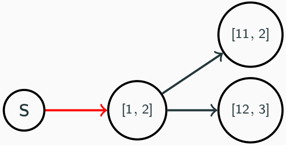

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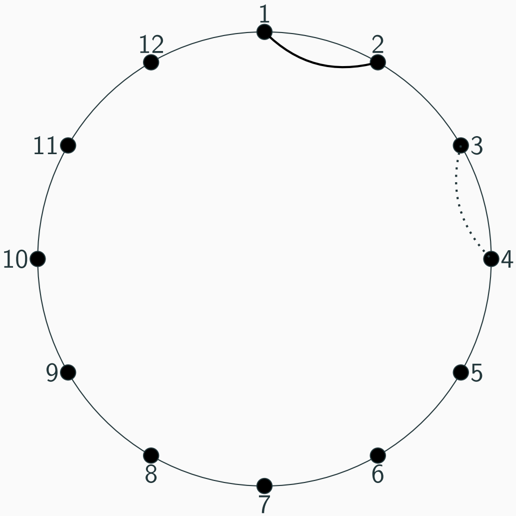

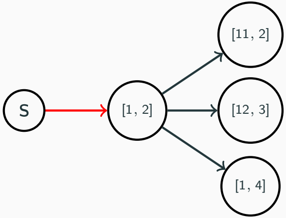

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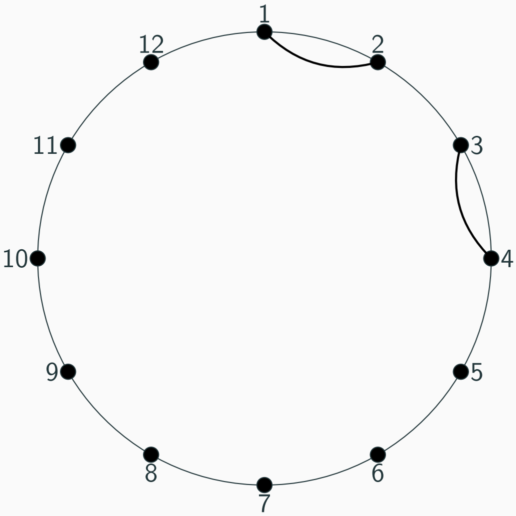

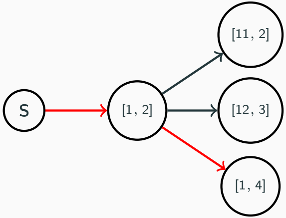

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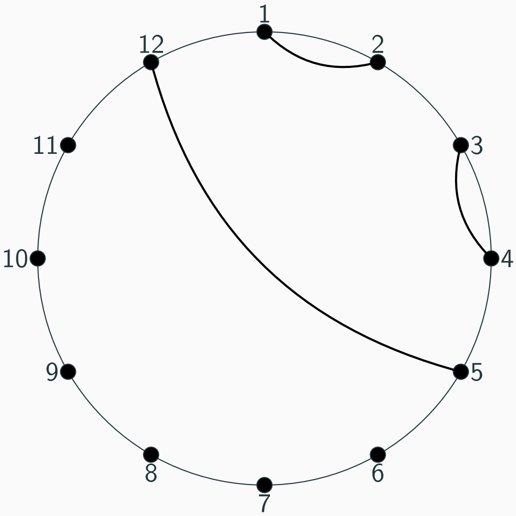

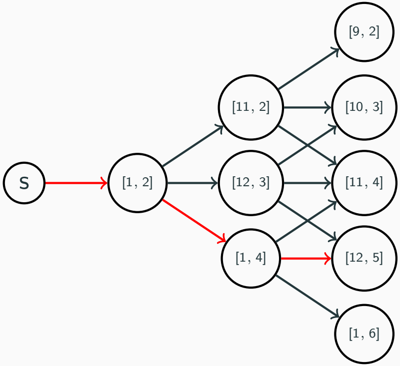

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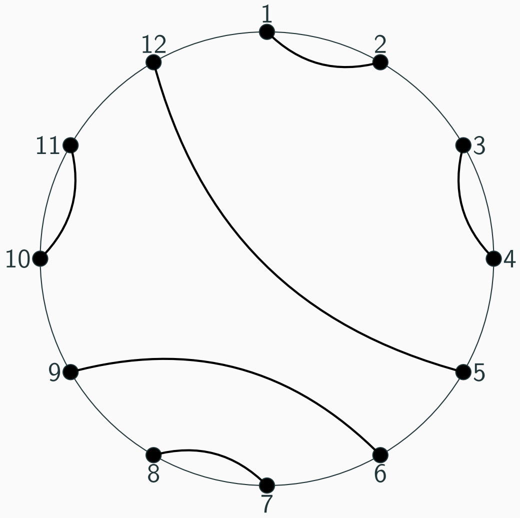

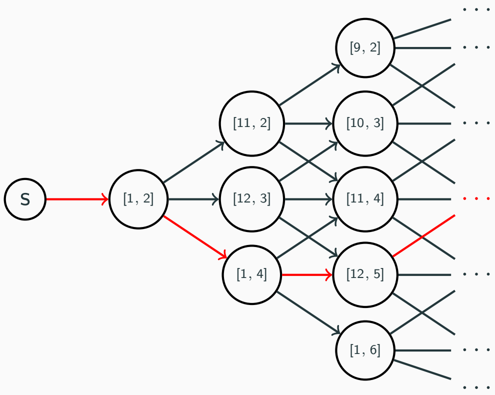

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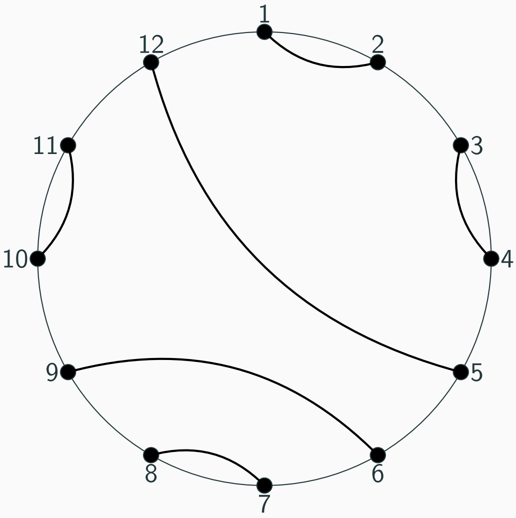
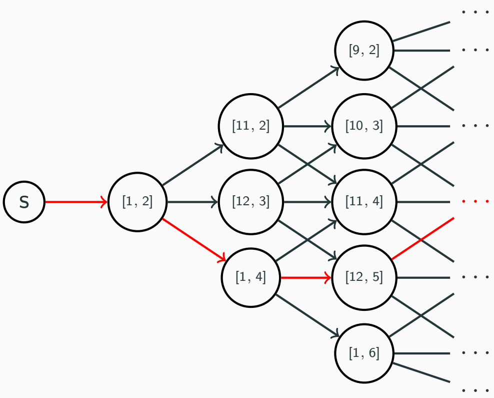


Every path corresponds to an element in $\mathbf{P}_{d/2}$.

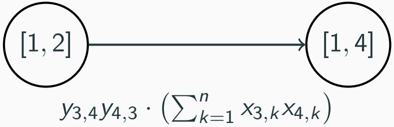


The Hard Polynomial

The Hard Polynomial

The new pair: $(3, 4)$.

The Hard Polynomial



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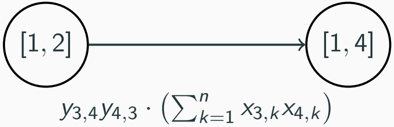

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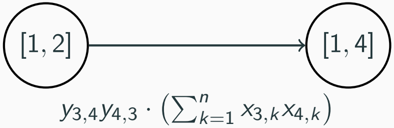
$(y_{3,4}y_{4,3})$: To select.

The Hard Polynomial



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$(y_{3,4}y_{4,3})$: To select.

$\left(\sum_{k=1}^{n} x_{3,k}x_{4,k}\right)$: To achieve full-rank.

The Hard Polynomial

$y_{3,4}y_{4,3} \cdot \left(\sum_{k=1}^{n} x_{3,k}x_{4,k}\right)$

	$x_{4,1}$	$x_{4,2}$	$\cdots$	$\cdots$	$x_{4,n}$
$x_{3,1}$	1	0	$\cdots$	$\cdots$	0
$x_{3,2}$	0	1	$\cdots$	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$			$\vdots$
$\vdots$	$\vdots$	$\vdots$			$\vdots$
$x_{3,n}$	0	0	$\cdots$	$\cdots$	1

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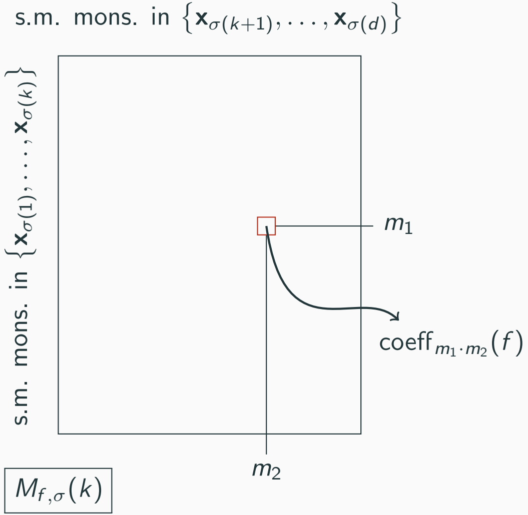
Lower Bound for a single osmABP



f is a set-multilinear poly. w.r.t $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$.

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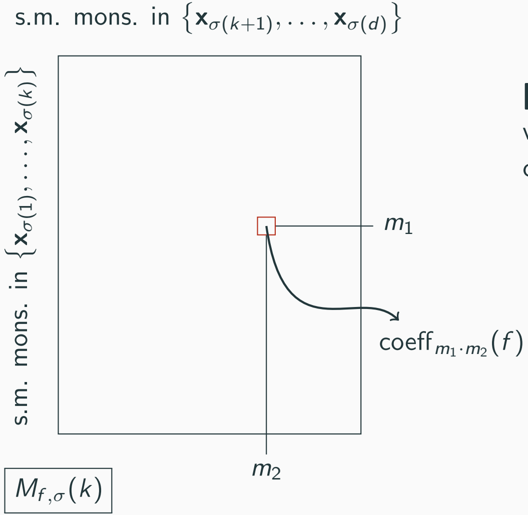

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[Nisan 91]: For every $1 \leq k \leq d$, the number of vertices in the k-th layer of the smallest osmABP(σ) computing f is equal to the rank of $M_{f,\sigma}(k)$.

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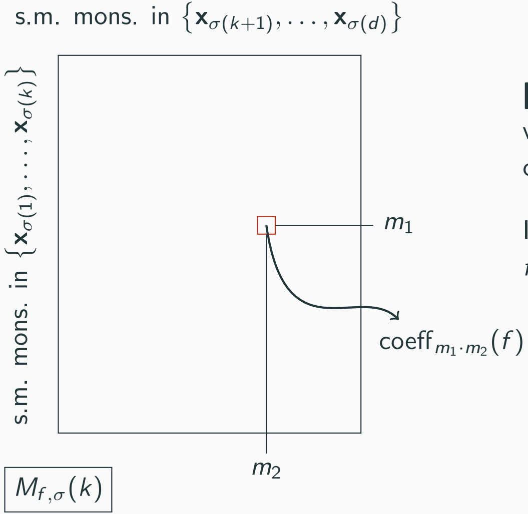

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If $\mathcal{A}$ is the smallest osmABP (in order σ) computing f, then

$$\text{size}(\mathcal{A}) = \sum_{i=1}^{d} \text{rank}(M_{f,\sigma}(k)).$$

Lower Bound for a single osmABP (contd.)

$$G_{n,d} = \sum_{\mathcal{P} \in \mathbf{P}_{d/2}} \prod_{(i,j) \in \mathcal{P}} y_{i,j} y_{j,i} \cdot \left(\sum_{k=1}^{n} x_{i,k} x_{j,k} \right).$$

Lower Bound for a single osmABP (contd.)

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Properties:

- $G_{n,d}$ is computable by a set-multilinear ABP of size poly(n, d).

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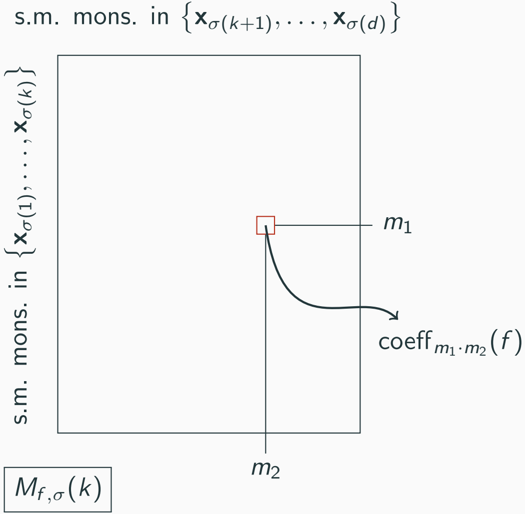

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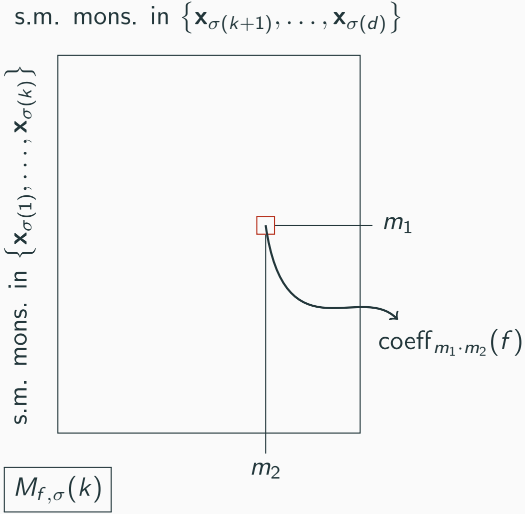
Lower Bound for a single osmABP (contd.)



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Therefore,

$$\text{rank}(M_{G_{n,d},\sigma}(d/2)) = \Omega(n^{d/8}).$$

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$$G_{n,d} = \sum_{i=1}^{t} g_i \quad \text{where} \quad g_i = \sum_{u_1,\ldots,u_{q-1}} \prod_{j=1}^{q} g^{(i)}_{u_{j-1},u_j}.$$

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for every i, w.h.p. there are many js, for which $M_w(g^{(i)}_{u_{j-1},u_j})$ is far from full rank

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$\implies$ for every i, w.h.p. $M_w(g_i)$ is far from full rank

Lower Bound for a Sum of osmABPs

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$\implies$ for every i, w.h.p. $M_w(g_i)$ is far from full rank

$\implies$ $M_w(G_{n,d})$ is far from full rank unless t is large.

Ongoing and Future Projects

Some Open Directions

- Better lower bounds against homogeneous formulas?

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- Super-linear Lower Bounds against ABPs for constant degree polynomials?

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- Better lower bounds against set-multilinear ABPs?
- Better Lower Bounds against Non-Commutative circuits?
- Separating formulas and ABPs in the non-commutative setting?
- Do VP have VP natural proofs under some reasonable conditions?

Branching Out

Study complexity theoretic questions about Boolean Circuits, Communication Models.

Teaching

The courses I would be happy to teach:

- Computing Lab -I
- Computing Lab-II
- Discrete Mathematical Structures
- Linear Algebra for Computer Science
- Programming and Data Structures
- Algorithms
- Theory of Computation
- Linear Programming and Convex Optimization

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Given enough time, I should be able to teach some of the other compulsory courses as well.

Thank you!