Lower Bounds for some Algebraic Models of Computation

Prerona Chatterjee

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Boolean Circuit Complexity: Given a boolean function f on n inputs, how many $\wedge$, $\vee$, $\neg$ gates are needed for a boolean circuit to compute f (in terms of n)?

Computing Formal Polynomials: Algebraic Models of Computation

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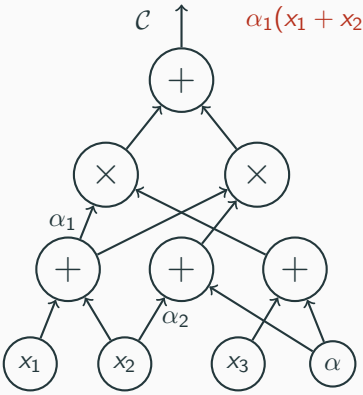

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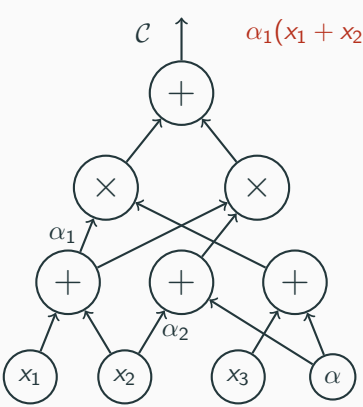

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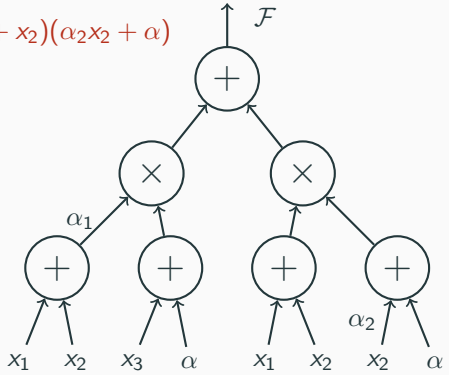

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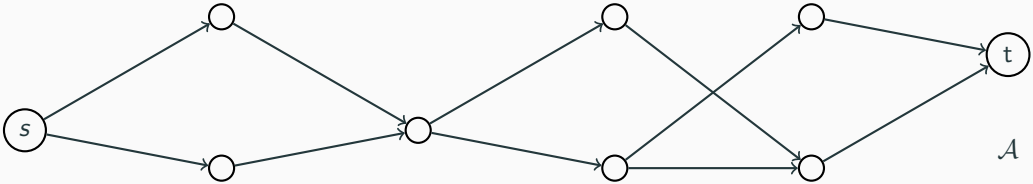

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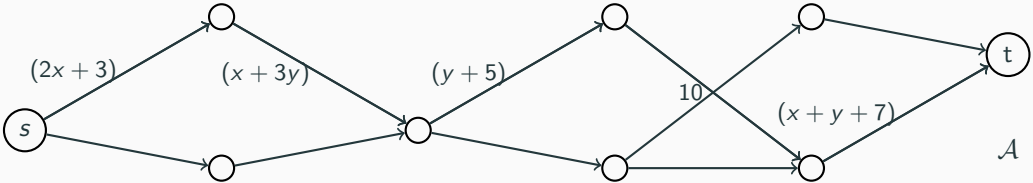

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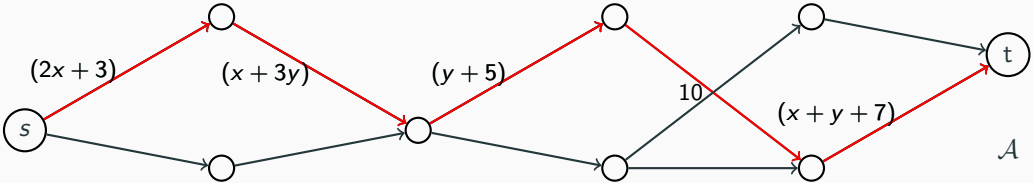

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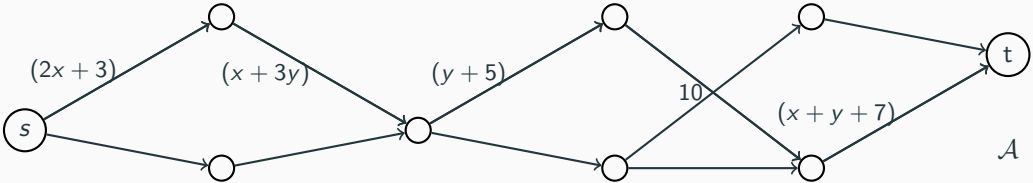

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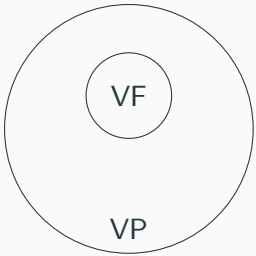

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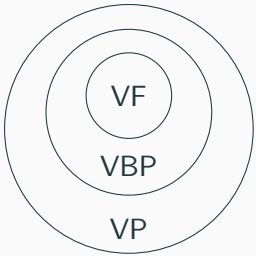

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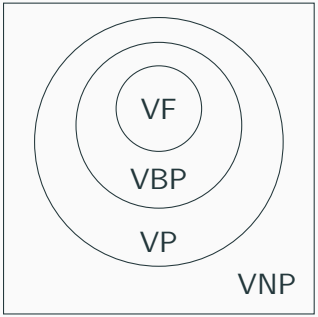

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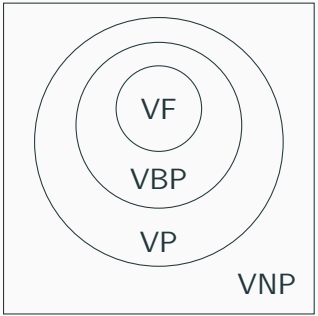

Central Question: Find explicit polynomials that cannot be computed by efficient circuits.

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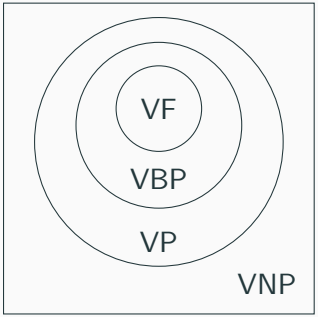

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[C-Kumar-She-Volk]: Any formula computing $\mathrm{ESYM}_{n,0.1n}(\mathbf{x})$ requires $\Omega(n^2)$ vertices, where

$$\mathrm{ESYM}_{n,d}(\mathbf{x}) = \sum_{i_1 < \cdots < i_d \in [n]} x_{i_1} \cdots x_{i_d}.$$

How does one make progress?

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Show that if a structured n-variate, degree-d polynomial is computable by a general model of size s, then they can also be computed by a structured model of size $\mathrm{func}(s, n, d)$ for some function func.

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A lot of work that culminated in

[Limaye-Srinivasan-Tavenas]

Any depth-3 or depth-4 circuit computing $\text{IMM}_{n,\log n}(\mathbf{x})$ must have size $n^{\Omega(\sqrt{d})}$.

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[C-Kush-Saraf-Shpilka]: For $\omega(\log n) = d \leq n$, there is a polynomial $G_{n,d}(\mathbf{x})$ which is set-multilinear w.r.t $\mathbf{x} = \{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$, where $|\mathbf{x}_i| \leq n$ for every $i \in [d]$, such that:

- $G_{n,d}$ is computable by a set-multilinear ABP of size poly(n),
- any $\sum$osmABP computing $G_{n,d}$ must have super-polynomial total-width.

Set-Multilinearity

The variable set is divided into buckets.

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An ABP is set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if every path in it

computes a set-multilinear monomial with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$.

Near Tightness of ABP Set-Multilinearisation

For $\sigma \in S_d$, an ABP is σ-ordered set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

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[C-Kush-Saraf-Shpilka]: Super polynomial lower bound against total-width of $\sum$osmABP for a polynomial of degree $d = \omega(\log n)$ that is computable by polynomial-sized ABPs.

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[Nisan]: Any ABP computing $\mathrm{Pal}_n(x_0, x_1) = \sum_{w \in \{0,1\}^{n/2}} \mathbf{x}_w \cdot \mathbf{x}_{w^R}$ has size $2^{\Omega(n)}$.

ABP vs Formula in the Non-Commutative Setting

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ABP vs Formula in the Non-Commutative Setting

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If an n-variate polynomial is abecedarian with respect to $\{X_1, \ldots, X_m\}$ for $m = \log n$, then any formula computing f can be made abecedarian with only $\mathrm{poly}(n)$ blow-up in size.

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