Lower Bounds Against Non-Commutative Models of Algebraic Computation

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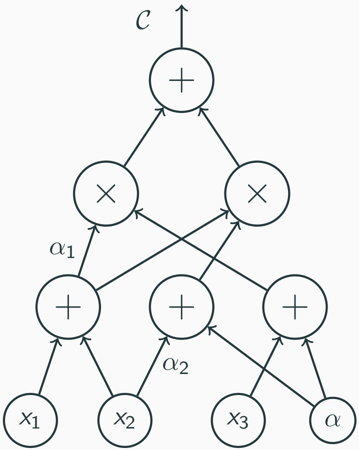
Model of interest today: Algebraic Circuits

Algebraic Circuits



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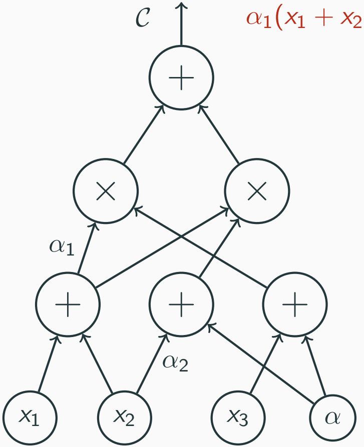

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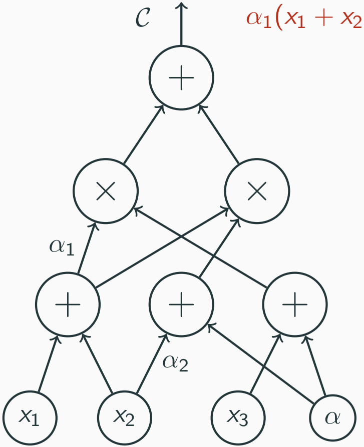

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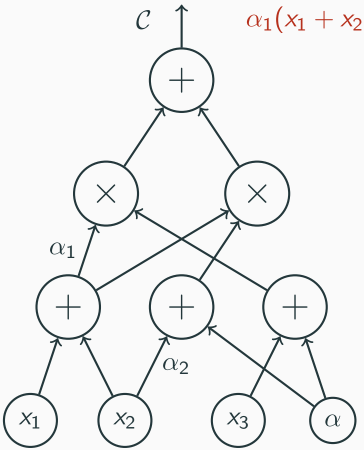

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Central Question: Find explicit polynomials that cannot be computed by circuits of size poly(n,d).

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Super-polynomial Lower Bound Against Constant Depth Circuits

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This is especially cool in the algebraic world.

Depth reduction results exist, which show that "good enough" super-polynomial lower bounds against constant depth circuits imply super-polynomial lower bounds against general circuits.

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Find an explicit polynomial that is hard!

The Non-Commutative Setting

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Can we do something better in this setting?

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Theorem: Any homogeneous non-commutative circuit computing

$$\mathrm{OSym}_{n,d} = \sum_{1 \le i_1 < \cdots < i_d \le n} x_{i_1} \cdots x_{i_d}$$

has size $\Omega(nd')$ where $d' = \min(d, n-d)$.

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Example: $f = x_1 \cdots x_d \implies f^{(0)} = x_1, \; f^{(d)} = x_d, \; f^{(i)} = x_i x_{i+1}$ for every $1 \leq i \leq d-1$.

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$\mathcal{C}$: Homogeneous non-commutative circuit.

$$\mu(\mathcal{C}) = \operatorname{rank}\left(\operatorname{span}_{\mathbb{F}}\left(\bigcup_{g\in\mathcal{C}}\left\{g^{(0)}, g^{(1)}, \ldots, g^{(d)}\right\}\right)\right).$$

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We already saw that for $f = x_1 \cdots x_d$, $\mu(f) = d+1$. Therefore $s \geq d$.

Using it to prove a “not so obvious” fact

Theorem: There exists an explicit monomial over $\{x_0, x_1\}$ of degree d such that any homogeneous non-commutative circuit computing it must have size $\Omega\left(\frac{d}{\log d}\right)$.

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In this case, if $\mathcal{C}$ is a homogeneous non-commutative circuit of size s, then $\mu_\ell(\mathcal{C}) \leq O(s \log d)$.

Therefore all we need is a monomial, f, over $\{x_0, x_1\}$ of degree d such that $\mu_\ell(f) \geq \Omega(d)$.

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Question: Can we prove the same lower bound against general non-commutative circuits?

Getting back to the main result

[Baur-Strassen]: If there is a circuit of size s computing $f \in \mathbb{F}[\mathbf{x}]$, then there is a circuit of size at most $5s$ that simultaneously compute $\{\partial_{x_1} f, \partial_{x_2} f, \ldots, \partial_{x_n} f\}$.

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- Suppose a similar result was true in the homogeneous non-commutative setting.
- Suppose there is an n-variate, degree-d polynomial f such that

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Note: $f = x_1 B_d(x_0^{(1)}, x_1^{(1)}) + \cdots + x_n B_d(x_0^{(n)}, x_1^{(n)})$ already (almost) has the required property.

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Note: f has a non-homogeneous non-commutative circuit of size $O(n \log^2 d)$.

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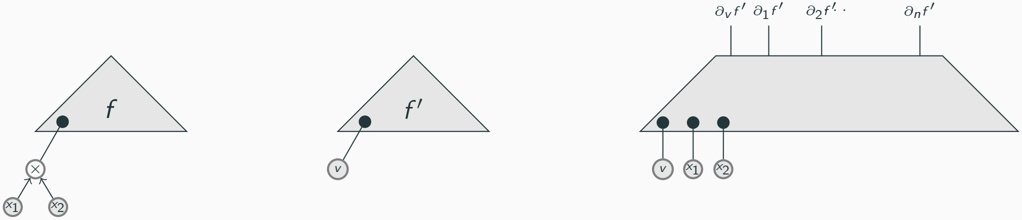

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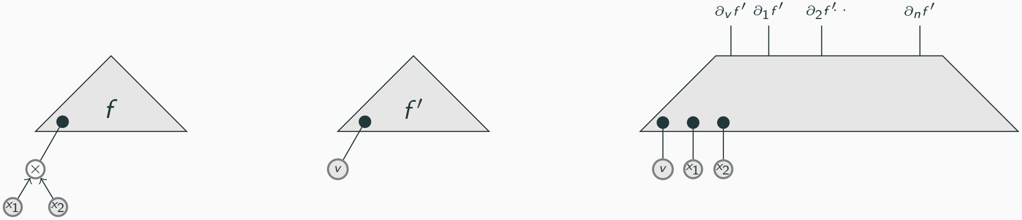

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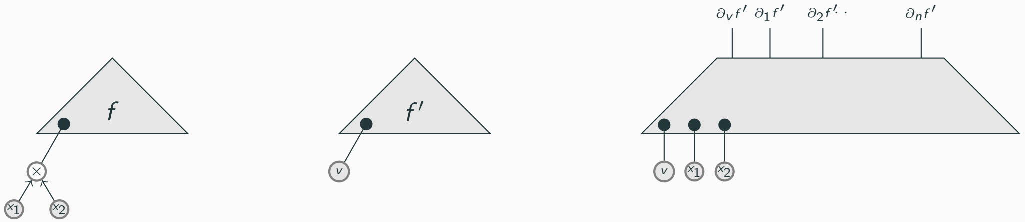

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Making [Baur-Strassen] work in the homogeneous setting

Target: If there is a homogeneous circuit of size s computing $f \in \mathbb{F}[\mathbf{x}]$, then there is a homogeneous circuit of size at most $5s$ that simultaneously compute $\{\partial_{x_1} f, \partial_{x_2} f, \ldots, \partial_{x_n} f\}$.

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Weights: $w_i = \mathrm{wt}(x_i)$. Given $\mathbf{w} = (w_1, \ldots, w_n)$, define $\mathbf{w}$-homogeneous.

Making [Baur-Strassen] work in the homogeneous setting

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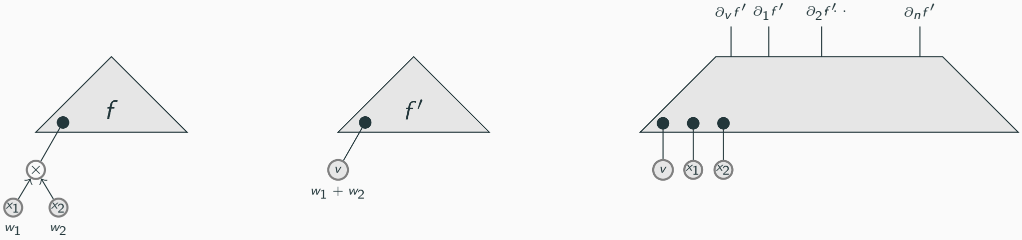

Making [Baur-Strassen] work in the non-commutative setting

Formal derivatives (with respect to the first position)

Given a polynomial f and a variable x, f can be uniquely written as

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where no monomial in f_1 contains x in the first position.

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Chain rules can be defined formally as well.

Lemma: If there is a homogeneous NC circuit of size s computing $f \in \mathbb{F}[\mathbf{x}]$, then there is a homogeneous NC circuit of size at most $5s$ that simultaneously compute $\{\partial_{1,x_1} f, \ldots, \partial_{1,x_n} f\}$.

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Task: Find n-variate, degree-d f such that if $\mathrm{out}(\mathcal{C}') = \{\partial_{1,x_1} f, \partial_{1,x_2} f, \ldots, \partial_{1,x_n} f\}$, then

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Use the fact that $\mu(\mathrm{out}(\mathcal{C}')) \leq \mu(\mathcal{C}')$ to complete the proof.

Recalling the measure and the polynomial

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$$\mu(f_1, \ldots f_n) = \text{rank}\left(\text{span}_{\mathbb{F}}\left(\bigcup_{i=1}^{n}\left\{f_i^{(0)}, f_i^{(1)}, \ldots, f_i^{(d)}\right\}\right)\right).$$

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The hard polynomial

$$\mathrm{OSym}_{n,\frac{n}{2}+1}(\mathbf{x}) = \sum_{1\le i_1<\cdots<i_{\frac{n}{2}+1}\le n} x_{i_1}x_{i_2}\cdots x_{i_{1+\frac{n}{2}}}$$

Polynomial with a large measure

$$f = \mathrm{OSym}_{n,\frac{n}{2}+1}(\mathbf{x}) = \sum_{1 \le i_1 < \cdots < i_{\frac{n}{2}+1} \le n} x_{i_1} x_{i_2} \cdots x_{i_{1+\frac{n}{2}}}$$

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$$f_i = \partial_{1,x_i} f = \sum_{i < i_1 < \cdots < i_{\frac{n}{2}} \le n} x_{i_1} x_{i_2} \cdots x_{i_{\frac{n}{2}}}$$

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Claim: The following set of size $\Omega(n^2)$ is linearly independent.

$$\left\{ f_i^{(j)} \;:\; 1 \le i \le \frac{n}{2}, \quad 0 < j < \frac{n}{2} \right\}.$$

Polynomial with a large measure

$x_{\frac{n}{2}+1}x_{\frac{n}{2}+2} \quad \cdots \quad x_2x_{\frac{n}{2}+2} \quad \cdots \quad \cdots \quad x_{n-2}x_{n-1} \quad \cdots \quad x_{\frac{n}{2}-1}x_{n-1} \quad x_{n-1}x_n \quad \cdots \quad x_{\frac{n}{2}}x_n$

$(1, \frac{n}{2})$

$\vdots$

$(1, 1)$

$\vdots$

$\vdots$

$(\frac{n}{2} - 2, \frac{n}{2})$

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$x_k x_l$

$\vdots$

$(\frac{n}{2} - 2, \frac{n}{2}) \qquad (j, i) \qquad \boxed{\text{coeff}_{x_k x_l}(f_i^{(j)})}$

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The matrix is lower triangular with the diagonal entries being all 1.

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This completes the proof of the main result.

The lower bound is tight

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Use the following fact recursively.

$$\mathrm{OSym}_{n,d}(x_1, \ldots, x_n) = \mathrm{OSym}_{n-1,d-1}(x_1, \ldots, x_{n-1}) \cdot x_n + \mathrm{OSym}_{n-1,d}(x_1, \ldots, x_{n-1}).$$

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There is a non-commutative circuit of size $O(n \log^2 n)$ that computes all the elementary symmetric polynomials simultaneously.

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Do polynomial multiplication recursively $\log n$ times. Note that polynomial multiplication can be done in time $O(n \log n)$ using FFT.

Open Questions

- Can we show a $\tilde{\Omega}(d)$ lower bound against general non-commutative circuits for a bivariate monomial of degree d?

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Conjecture: If

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can be computed by a non-commutative circuit of size s, then $\{f_1, \ldots, f_d\}$ can be simultaneously computed by a non-commutative circuit of size $d + O(s)$.

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If true, then the answer to the second question is "yes".

Hardness Amplification

[Carmossino-Impagliazzo-Lovett-Mihajlin]: Super-linear lower bounds $\left(n^{\Omega\left(\frac{\omega}{2}+\varepsilon\right)}\right)$ against non-commutative circuits for constant degree polynomials imply exponential lower bounds.

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Question: Can we show a similar statement (or any non-trivial hardness amplification statement) in the non-constant degree setting?

Open Questions

Thank you!