

IPS Lower Bounds for Formulas and Sum of ROABPs

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based on work with

Utsab Ghosal (Imperial College), Partha Mukhopadhyay & Amit Sinhababu (CMI)

Ideal Proof System

Given: $f_1, \dots, f_k \in \mathbb{F}[\bar{x}]$

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(preferably from CNF instances)

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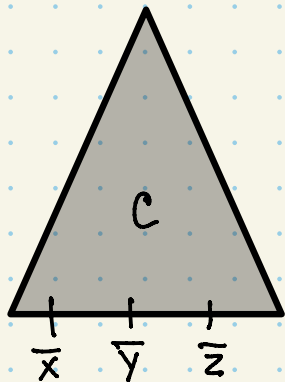
(proving the unsatisfiability of the CNF instances)

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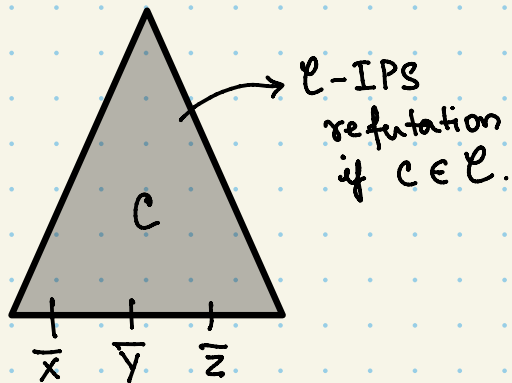
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FSTW: For $\mathcal{C} \equiv$ poly-sized ckt's / formulas, super-poly l.b. against $\mathcal{C}$ -IPS_{Lin} $\Rightarrow$ super-poly. l.b. against $\mathcal{C}$ -IPS.

Revised Goal: Prove a lower bound against every polynomial g that agrees with $1/f$ on $\{0,1\}^n$.

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FSTW: Proved such super-poly. lower bounds over char-0

fields for

- $\mathcal{C} \equiv$ Sparse polynomials
- $\mathcal{C} \equiv \Sigma\Lambda\Sigma$ circuits
- $\mathcal{C} \equiv$ ROABPs
- $\mathcal{C} \equiv$ Multilinear formulas
- $\mathcal{C} \equiv$ Constant-depth multilinear formulas

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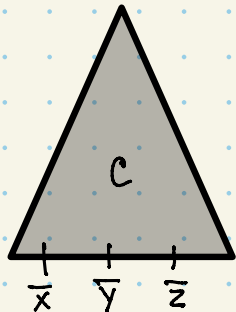
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i.e. Can we show FSTW type result for
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Issue : Not clear if the proof technique can be used to show functional lower bounds.

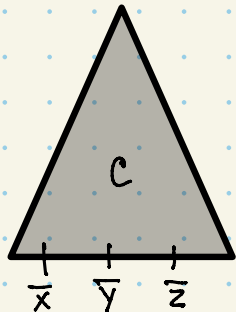
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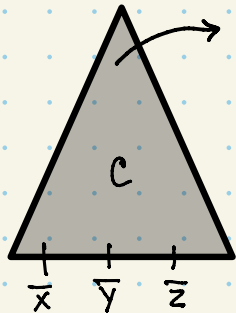
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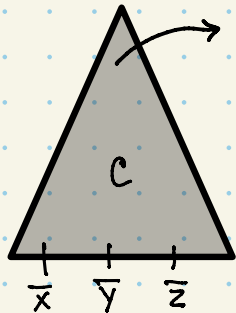
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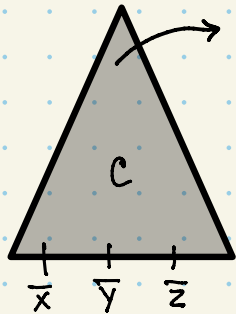
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GHT: Super-polynomial lower bound over char. 0 against
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Super-poly lower bound against constant depth algebraic
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Govindasamy-Hakoniemi-Tzameret & Hakoniemi-Limaye-Tzameret

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Super-poly lower bound against constant depth algebraic circuits via a different technique.

Hakoniemi - Limaye - Tzameret: Used the result of AGKT to prove the required functional lower bound.

HLT 24: Super-polynomial lower bound against $\mathcal{P}$ -IPS
(Char. 0) for $\mathcal{C} \equiv$ constant depth algebraic circuits when
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Other examples

FSTW: Method of multiples

Andrews-Forbes:
Lower bound against
Const. depth-IPS.

Finite Characteristic Setting

Behara - Limaye - Ramanathan - Srinivasan

Extended these results to fields of finite characteristic but large size. $\rightarrow$ Prev. Talk.

Finite Characteristic Setting

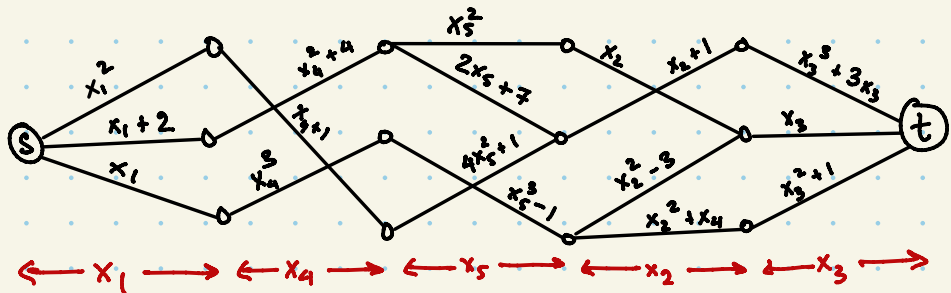
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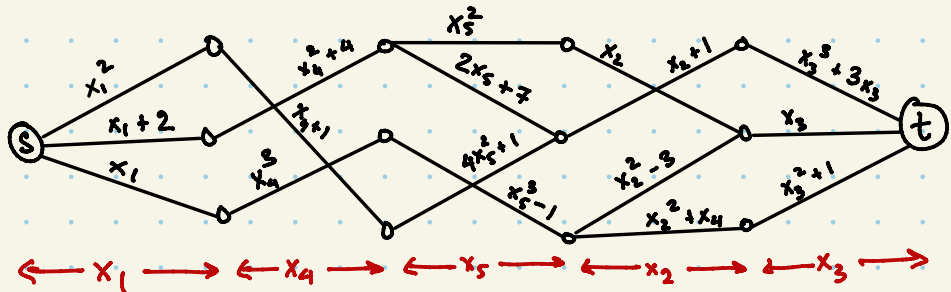
Elbaz - Govindasamy - Lu - Tzameret

Proved super-polynomial lower bound against mult-const. depth- IPS_{lin} over finite fields.

Sum of ROABPs

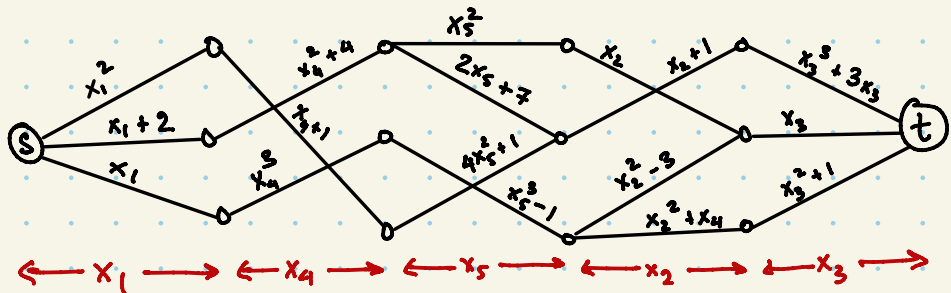


Sum of ROABPs



Order : x_1, x_4, x_5, x_2, x_3

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Sum of ROABPs: Sum of ROABPs with diff. orders.

Sum of ROABP Lower Bound

C - Kush - Saneef - Shpilka : Exponential Lower bound on
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Our Result : Exponential Lower Bounds against $\mathcal{C}$ -IPS_{Lin} where $\mathcal{C} \equiv$ Sum of ROABPs.

Sum of ROABPs IPS Lower Bound

One Result: Exponential Lower Bounds against $\mathcal{C}$ -IPS_{Lin},
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Theorem 1.6. For $X = \{x_0, \dots, x_{2n-1}\}$, $T = \{t_{i,j} : i, j \in [0, \dots, 2n-1] \text{ with } i < j\}$ and $\beta = 2^{\binom{2n}{2}}$, let $f \in \mathbb{F}[X, T]$ be the polynomial defined as

$$f = \left(\sum_{0 \leq i < j \leq 2n-1} t_{i,j} x_i x_j \right) - \beta$$

which is unsatisfiable over the Boolean hypercube. Then there exists $\gamma > 0$ such that the total width of any sum of ROABP computing the linear proof of unsatisfiability (IPS_{Lin}), for the system $\{f = 0, X^2 - X = 0, T^2 - T = 0\}$, is at least $\exp(n^\gamma)$.

→ Over Char 0. fields
(combining ideas from CKSS and FSTW)

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Theorem 4.11. Let $n \in \mathbb{N}$ and p be any prime. Further let $\mathbb{F}'$ be a field of characteristic p with size p^{k+1} , where k is the smallest integer such that $p^k > \binom{2n}{2} 2^{2n}$ and $\mathbb{F} \subset \mathbb{F}'$ be the subfield of size p^k . Also, arbitrarily fix $\beta \in \mathbb{F}' \setminus \mathbb{F}$.

For $X = \{x_0, \dots, x_{2n-1}\}$, $T = \{t_{i,j} : i, j \in \{0, \dots, 2n-1\} \text{ with } i < j\}$ and any $\tilde{\alpha} \in \mathbb{F}'^{\binom{2n}{2}}$, define

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Theorem 4.13. Let $\mathbb{F}$ be a field of characteristic at least 5. Further, let $X = \{x_1, \dots, x_{4n}\}$ and $T = \{t_{i,j,k,l} : i, j, k, l \in [4n] \text{ with } i < j < k < l\}$ be two sets of variables.

Define $g \in \mathbb{F}[X, T]$ to be the polynomial

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Over finite field (adding ideas from HLT) ←

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Over finite field (adding ideas from HLT)
 in place holder model.

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Can we still prove
matching LBs in
the proof setting?

Algebraic Formulas

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Q: Can these lower bounds be extended to the proof complexity setting?

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Main Lemma: Size of any formula computing f has size at least

$$\sum_{i=1}^k \text{alg-rank}(\{f_{i,m} : m \in \text{mon}(X_i)\})$$

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Obs: $f_1, \dots, f_k$ are algebraically dependent
 $\Rightarrow$ trailing monomials of $f_1, \dots, f_k$ are algebraically dependent.

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Need: f s.t. if g is mult and $g \equiv 1/f$ over $\{0,1\}^n$,
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To Do: Study the relation between the Support
structures of f and g .

Algebraic Formula IPS Lower Bound

Key Lemmas :

g : mult k

$g \equiv 1/f$ over $\{0,1\}^n$

- ① For "nice" f s, $\text{supp}(f) \subseteq \text{supp}(g)$
- ② Any $m \in \text{supp}(g) \setminus \text{supp}(f)$ is a multilinearised product of monomials in $\text{supp}(f)$.

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- ② Any $m \in \text{supp}(g) \setminus \text{supp}(f)$ is a multilinearised product of monomials in $\text{supp}(f)$.

g : mult $\wedge$
 $g \equiv 1/f$ over $\{0,1\}^n$

Choosing f to be a variant of the Subset-Sum axiom polynomial gives us the reqd. statement.

$$f'(\bar{x}, \bar{y}) = \sum_{i=1}^N \sum_{S \subseteq X_i} \left(\prod_{j \in S} x_j \cdot y_{t(S)} \right).$$

Hard CNF instances for ROABP IPS

Hakonien - Limaye - Tzameret (2026)

Theorem 1 (Main lower bound (informal); see [Theorem 18](#)). *There is a family of CNF formulas Ψ_n in $\text{poly}(n)$ variables and $\text{poly}(n)$ clauses that requires $\text{roABP-IPS}_{\text{LIN}}$ refutations of size $2^{n^{\Omega(1)}}$ in any variable ordering, and over any field.*

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Small ROABP
computing the
refutation of Ψ_n

$\Rightarrow$

Small monotone span
program computing the
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of Ψ_n .

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 - Monotone Circuits
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Thank you !!