Understanding the Power of Algebraic Models of Computation

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Complexity Theory

Addition v.s. Multiplication v.s. Factorisation

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Class I v.s. Class II v.s. Class IV/V

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- design a computational model that captures the constraints
- study the amount of resource required by the model to complete the task.

Complexity of Computing Polynomials

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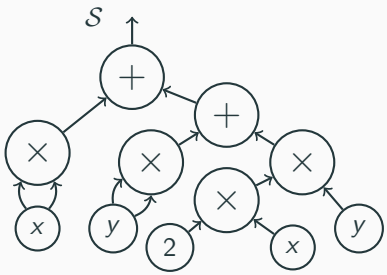

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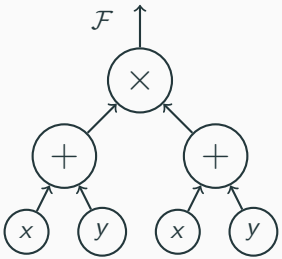

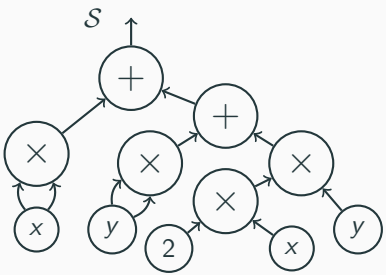

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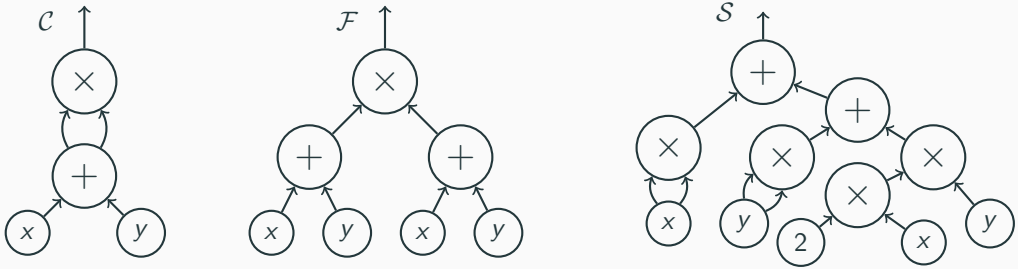

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- **[Shamir 79, Lipton 94]**: If $h(x) = \prod_{i=1}^{d}(x - i)$ can be computed using $\mathrm{poly}(\log d)$ additions and multiplications, then integer factoring is easy for boolean circuits.

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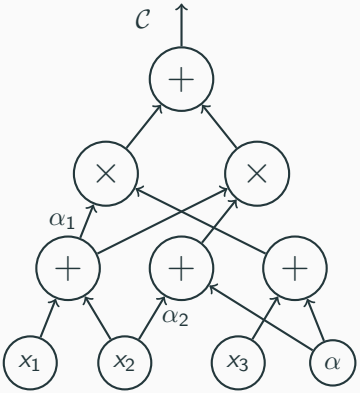

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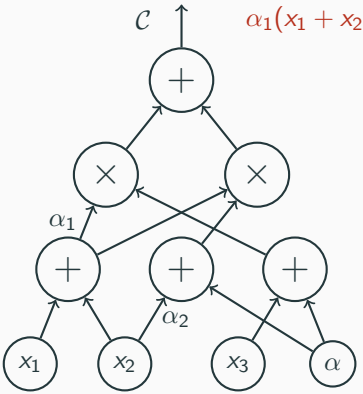

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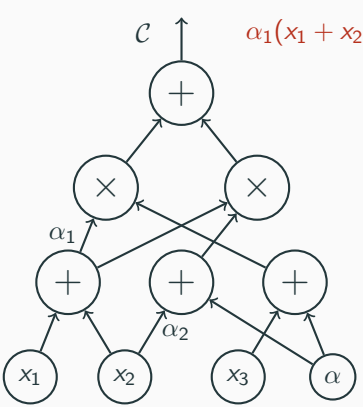
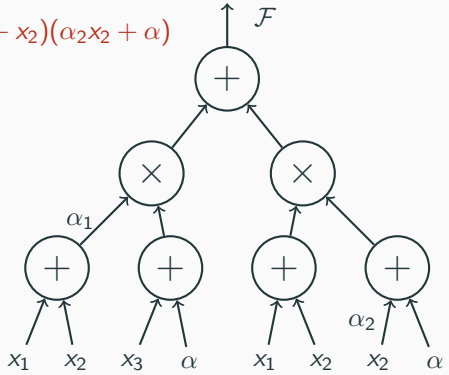
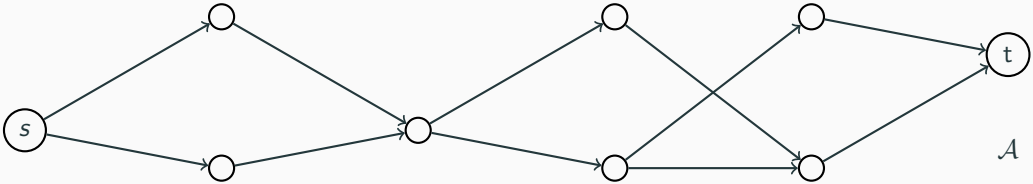


$$\alpha_1(x_1 + x_2)(x_3 + \alpha) + (x_1 + x_2)(\alpha_2 x_2 + \alpha)$$



Algebraic Branching Programs



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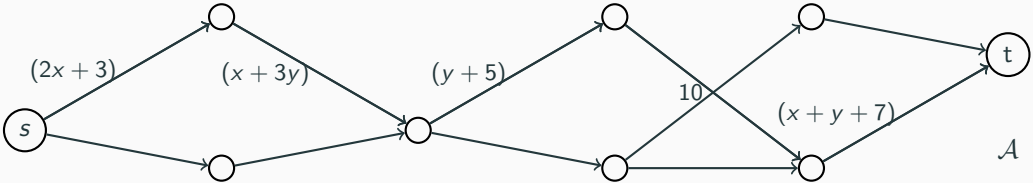

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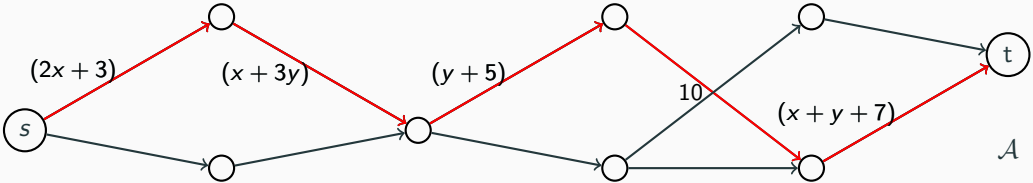

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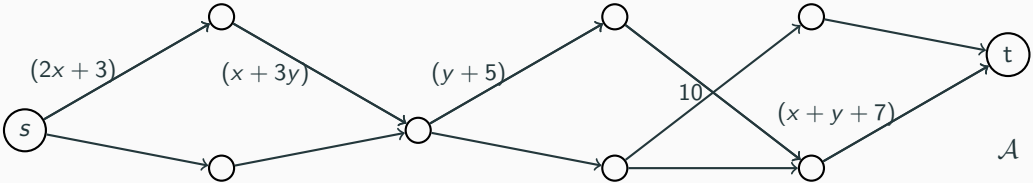

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- Polynomial computed by the ABP: $f_{\mathcal{A}}(\mathbf{x}) = \sum_p \mathrm{wt}(p)$

Iterated Matrix Multiplication

$$\begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix}^T \begin{bmatrix} \ell_{11}^{(1)} & \cdots & \ell_{1j}^{(1)} & \cdots & \ell_{1w}^{(1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \ell_{i1}^{(1)} & \cdots & \ell_{ij}^{(1)} & \cdots & \ell_{iw}^{(1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \ell_{w1}^{(1)} & \cdots & \ell_{wj}^{(1)} & \cdots & \ell_{ww}^{(1)} \end{bmatrix} \cdots\cdots \begin{bmatrix} \ell_{11}^{(d)} & \cdots & \ell_{1j}^{(d)} & \cdots & \ell_{1w}^{(d)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \ell_{i1}^{(d)} & \cdots & \ell_{ij}^{(d)} & \cdots & \ell_{iw}^{(d)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \ell_{w1}^{(d)} & \cdots & \ell_{wj}^{(d)} & \cdots & \ell_{ww}^{(d)} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

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Canonical example of a polynomial that is computable by an ABP of width w and length d.

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Complete for the class of all explicit polynomials.

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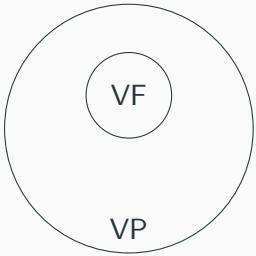

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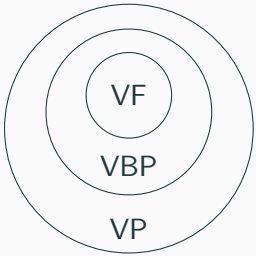

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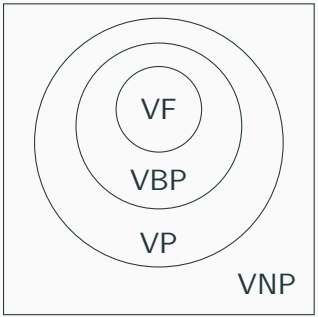

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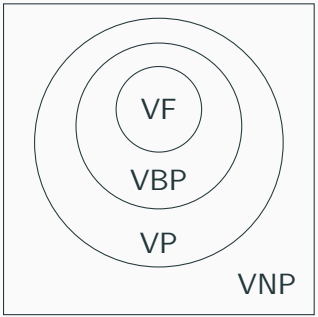

Central Question: Find explicit polynomials that cannot be computed by efficient circuits.

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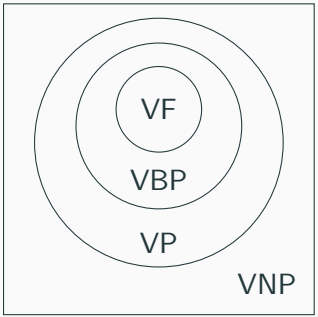
VNP: Explicit Polynomials

$$\text{VP} \stackrel{?}{=} \text{VNP}$$



Central Question: Find explicit polynomials that cannot be computed by efficient circuits.

Lower Bounds for General Models

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[C-Kumar-She-Volk 22]: Any formula computing $\mathrm{ESYM}_{n,0.1n}(\mathbf{x})$ requires $\Omega(n^2)$ vertices.

How does one make progress?

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Ultimate Goal: Prove better than $\text{func}(s, n, d)$ lower bounds.

Super-Polynomial Lower Bound against Constant Dept Circuits

[Limaye-Srinivasan-Tavenas 21] (JACM 2025)

Let d, n be such that $d \leq \frac{\log n}{100}$. For any $\Delta > 0$, any product-depth $\Delta \geq 1$ circuit computing $\mathrm{IMM}_{n,d}$ over any field of characteristic zero or $\geq d$, requires size $n^{\Omega\left(d^{\frac{1}{\exp(\Delta)}}\right)}$.

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[Forbes 24]: The theorem is true over all fields.

Set-Multilinearity

The variable set is divided into buckets.

$$\mathbf{x} = \mathbf{x}_1 \cup \cdots \cup \mathbf{x}_d \quad \text{where} \quad \mathbf{x}_i = \{x_{i,1}, \ldots x_{i,n_i}\}.$$

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f is set-multilinear with respect to $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ if

every monomial in f has exactly one variable from $\mathbf{x}_i$ for each $i \in [d]$.

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Step 1: Convert any arbitrary product-depth Δ circuit into a homogeneous product-depth 2Δ circuit computing the same polynomial. Blow-up in size is $2^{O(\sqrt{d})}\,\mathrm{poly}(s)$.

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Step 3: Prove a $n^{\Omega\left(\frac{d^{1/2^{\Delta}}-1}{\Delta}\right)}$ lower bound against set-multilinear constant depth circuits.

Questions?

Algebraic Branching Programs

$$\begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix}^T \begin{bmatrix} \ell_{11}^{(1)} & \cdots & \ell_{1j}^{(1)} & \cdots & \ell_{1w}^{(1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \ell_{i1}^{(1)} & \cdots & \ell_{ij}^{(1)} & \cdots & \ell_{iw}^{(1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \ell_{w1}^{(1)} & \cdots & \ell_{wj}^{(1)} & \cdots & \ell_{ww}^{(1)} \end{bmatrix} \cdots\cdots \begin{bmatrix} \ell_{11}^{(d)} & \cdots & \ell_{1j}^{(d)} & \cdots & \ell_{1w}^{(d)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \ell_{i1}^{(d)} & \cdots & \ell_{ij}^{(d)} & \cdots & \ell_{iw}^{(d)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \ell_{w1}^{(d)} & \cdots & \ell_{wj}^{(d)} & \cdots & \ell_{ww}^{(d)} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

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What is the smallest w for which the polynomial of interest can be computed as above?

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Our Result

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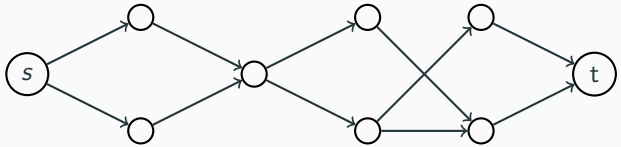
At most d matrices: Proof Overview



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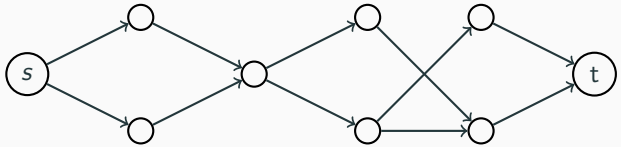

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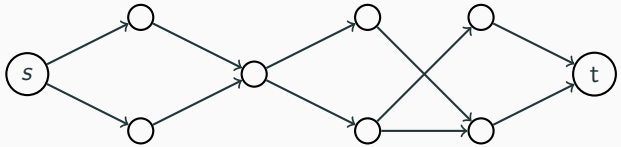

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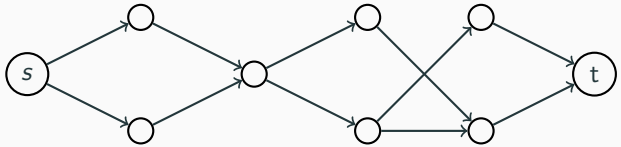

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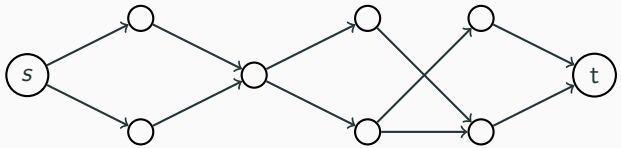

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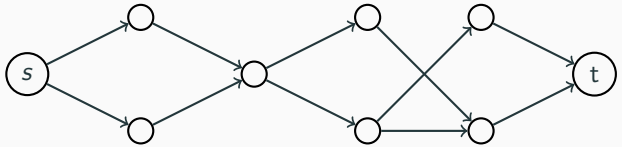

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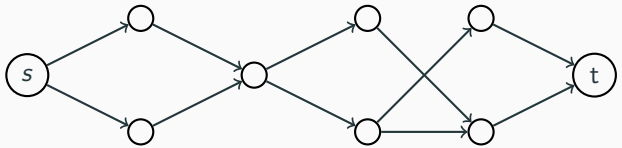

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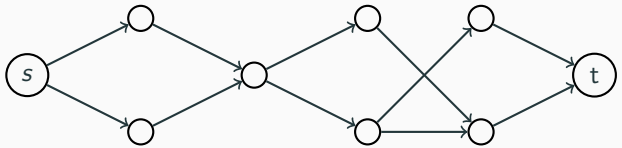

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Note: $\deg(g_i'), \deg(h_i') \leq [d-1]$ for every $i \in [d-1]$. Thus $\deg(R) \leq d-1$.

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Look at the first order derivatives.

$$S = \{\partial_{x_i}(P_{n,d} - R)\} = \left\{d \cdot x_i^{d-1} - \partial_{x_i}(R)\right\}_{i \in [n]}$$

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At most d matrices: Proof Overview (contd.)

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Proof Overview Of Our Result

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$((n/2) - r) \cdot (d-1)$ vertices.

Proof Overview Of Our Result (contd.)

Step 2: Iteratively reduce to Base Case

In each iteration, reduce the number of layers till it becomes $(d+1)$ such that

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In each iteration, reduce the number of layers till it becomes $(d+1)$ such that

- the number of layers is reduced by a constant fraction,
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$$f_{\ell+1} = \sum_{i=1}^{n} x_i^d + \sum_{i=1}^{r_{\ell+1}} A_i(\mathbf{x}) \cdot B_i(\mathbf{x}) + \delta_{\ell+1}(\mathbf{x})$$

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- number of error terms collected is small.

The Induction Step

ℓ-th step

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Given: $\mathcal{A}_\ell$

Size $= s_\ell$
Number of layers $= d_\ell$
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Want to construct: $\mathcal{A}_{\ell+1}$

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Number of Error Terms: $\varepsilon \cdot n$ where ε can be chosen.

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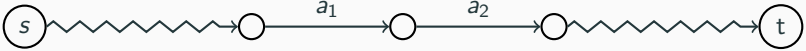
The Lower Bound: $((n/2) - \varepsilon \cdot n) \cdot (d-1)$

Proof of the Induction Step



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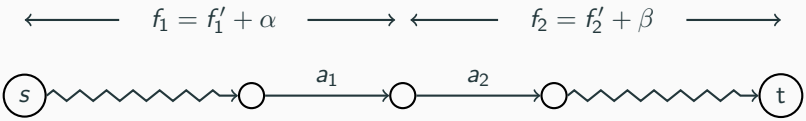
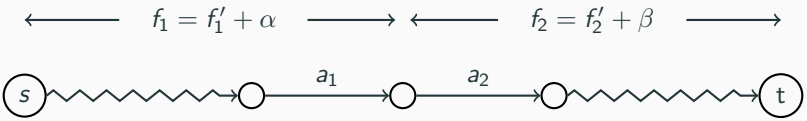


Proof of the Induction Step

$\mathcal{A}_\ell = f_1 \cdot f_2$



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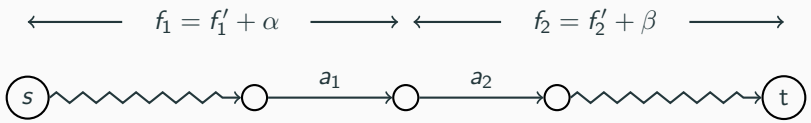

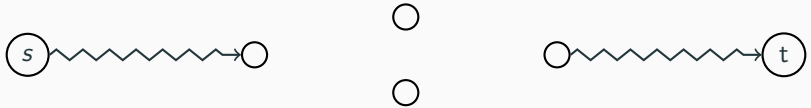

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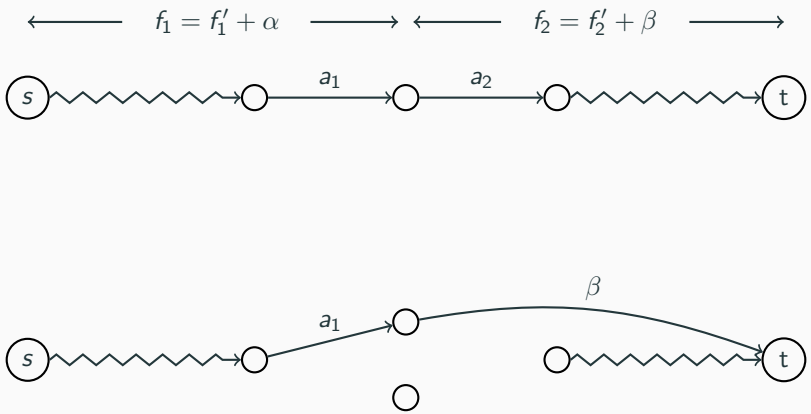

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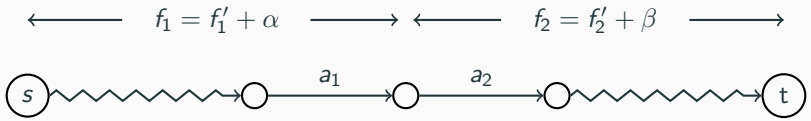

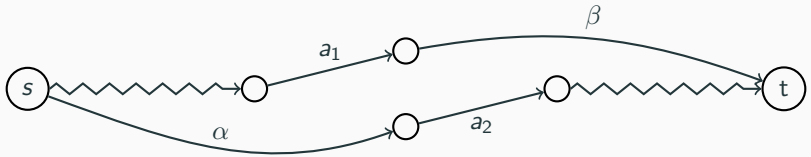

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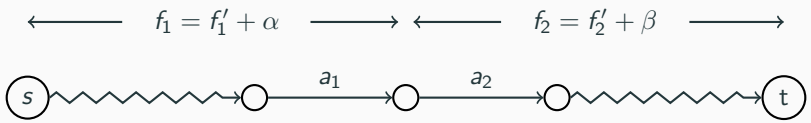

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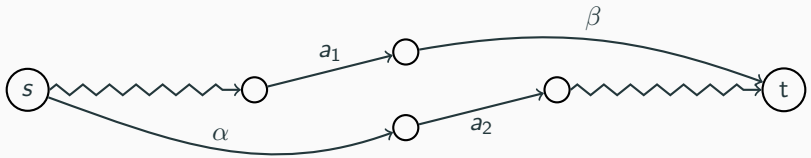

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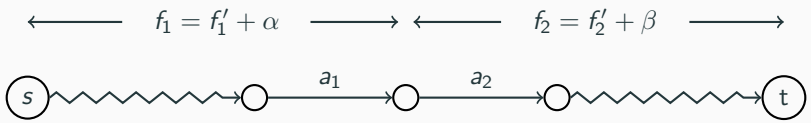
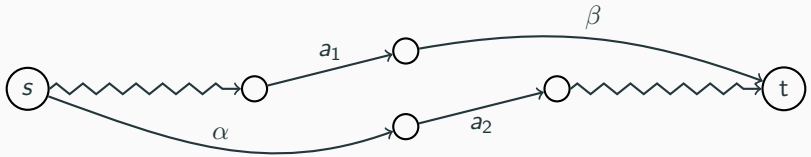


$\mathcal{A}_{\ell+1} = \beta \cdot f_1 + \alpha \cdot f_2 = \mathcal{A}_\ell - f_1' \cdot f_2' + \alpha \cdot \beta$



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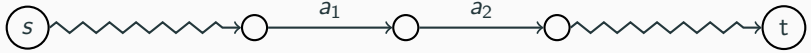

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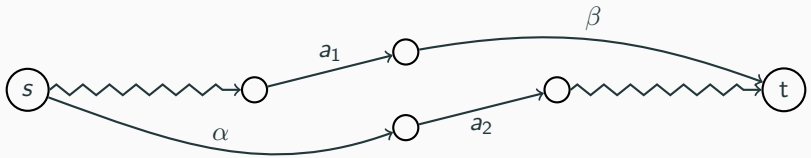

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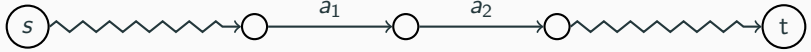

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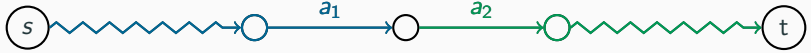

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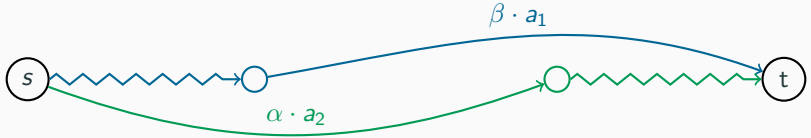

Thank you!!!