Weekly Problems: 1.

Due: 2pm on 11/08/25

Instructions

- Discussion is allowed and infact encouraged
- Answers must be written by yousef.
- All sources that are used to reach the solution must be mentioned.

1. Find the error in the following proof that all horses are the same colour.

CLAIM: In any set of h horses, all horses are the same colour.

Proof: By induction on h.

Base Case: $h = 1$.

In any set containing just one horse, all horses are clearly of the same colour.

Induction Step: For $k \geq 1$, assuming the claim is true for $h = k$ and proving for $h = k+1$

Take any set H of $k+1$ horses. We will show that all the horses in this set are of the same colour. Remove one horse from this set to obtain the set H_1 with just k horses. By the induction hypothesis, all the horses in H_1 are of the same colour. Now replace the removed horse and remove a different horse to obtain the set H_2. By the same argument, all the horses in H_2 are the same colour. Therefore all the horses in H must be the same colour.

[2]

2. Let A, B, C be sets with a, b, c many elements respectively.

i) How many elements does $A \times B$ have?

ii) How many elements does the power set of C have?

[2 + 2]

(3) Consider the undirected graph $G = (V, E)$

where V, the set of nodes, is $\{1, 2, 3, 4\}$
and E, the set of edges, is $\{\{1,2\}, \{2,3\},$
$\{1,3\}, \{2,4\}, \{1,4\}\}$.

i) Draw the graph G.
ii) What are the degrees of each node?
iii) Indicate a path from node 3 to node 4 on your drawing of G.

[1 + 2 + 1]

(4) Write a logical expression $\varphi(x)$ involving the variables x_0, x_1, x_2 and the operators AND ($\wedge$), OR ($\vee$), NOT ($\neg$) such that $\varphi(x)$ is true if the majority of the inputs are true.

[3]

(5) Use logical quantifiers ($\forall, \exists$), logical operators ($\vee, \wedge, \neg$), arithmetic operations ($+, \times, =, >, <$) to write an expression $\varphi(n,k)$ s.t. for every natural no. n, k, $\varphi(n,k)$ is

true iff k divides n [2]

(6) Prove or disprove:

Let S be the set of all functions mapping $\{0,1\}^n$ to $\{0,1\}$. Then, there is an injective map from S to $T = \{0,1\}^{n^3}$. Here assume that $n \geq 5$. [2].

(7) Let A, B be two finite sets. Prove that $|A \cup B| = |A| + |B| - |A \cap B|$. [3].
