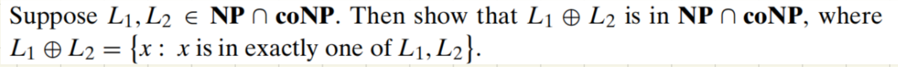
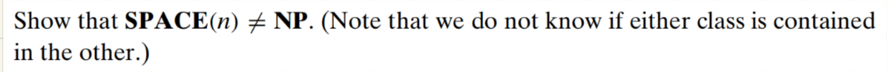
Assignment 2

[Due : 13th April 5:00 pm]

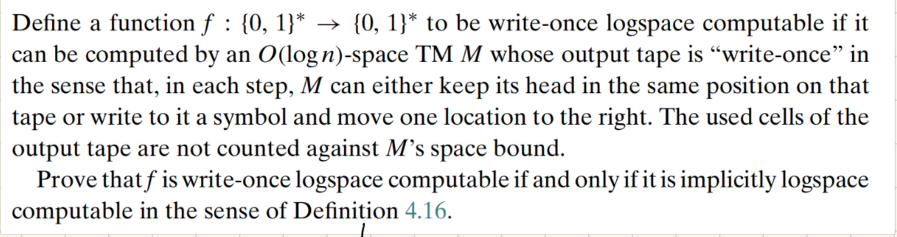
① 
Suppose $L_1, L_2 \in \mathbf{NP} \cap \mathbf{coNP}$. Then show that $L_1 \oplus L_2$ is in $\mathbf{NP} \cap \mathbf{coNP}$, where $L_1 \oplus L_2 = \{x : x \text{ is in exactly one of } L_1, L_2\}$. [8]

② 
Show that $\mathbf{SPACE}(n) \neq \mathbf{NP}$. (Note that we do not know if either class is contained in the other.) [4]

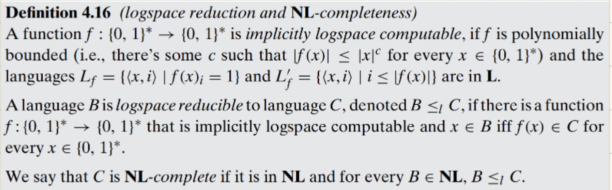
Hint: Space Hierarchy Theorem

③ Show that the complement of every NP-complete language is co-NP complete [3]

④ Show that $\Sigma_2 \cap \Pi_2$ is closed under taking union [i.e. $L_1, L_2 \in \Sigma_2 \cap \Pi_2 \Rightarrow L_1 \cup L_2 \in \Sigma_2 \cap \Pi_2$] [4]

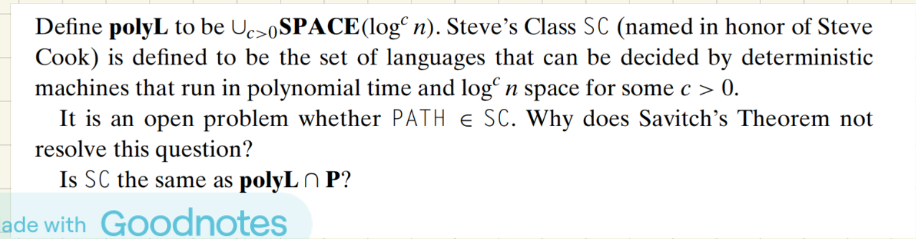
⑤ 
Define a function $f : \{0,1\}^* \to \{0,1\}^*$ to be write-once logspace computable if it can be computed by an $O(\log n)$-space TM M whose output tape is "write-once" in the sense that, in each step, M can either keep its head in the same position on that tape or write to it a symbol and move one location to the right. The used cells of the output tape are not counted against M's space bound.

Prove that f is write-once logspace computable if and only if it is implicitly logspace computable in the sense of Definition 4.16. [6]


Definition 4.16 *(logspace reduction and* **NL***-completeness)*
A function $f : \{0,1\}^* \to \{0,1\}^*$ is *implicitly logspace computable*, if f is polynomially bounded (i.e., there's some c such that $|f(x)| \leq |x|^c$ for every $x \in \{0,1\}^*$) and the languages $L_f = \{\langle x, i\rangle \mid f(x)_i = 1\}$ and $L'_f = \{\langle x, i\rangle \mid i \leq |f(x)|\}$ are in $\mathbf{L}$.

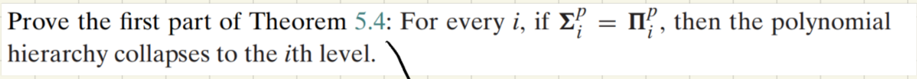
A language B is *logspace reducible* to language C, denoted $B \leq_l C$, if there is a function $f : \{0,1\}^* \to \{0,1\}^*$ that is implicitly logspace computable and $x \in B$ iff $f(x) \in C$ for every $x \in \{0,1\}^*$.

We say that C is $\mathbf{NL}$*-complete* if it is in $\mathbf{NL}$ and for every $B \in \mathbf{NL}$, $B \leq_l C$.

⑥ 
Define **polyL** to be $\cup_{c>0}\mathbf{SPACE}(\log^c n)$. Steve's Class SC (named in honor of Steve Cook) is defined to be the set of languages that can be decided by deterministic machines that run in polynomial time and $\log^c n$ space for some $c > 0$.

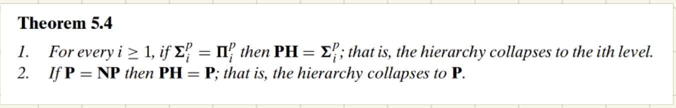
It is an open problem whether PATH $\in$ SC. Why does Savitch's Theorem not resolve this question?

Is SC the same as **polyL** $\cap$ **P**? [4]

⑦ 

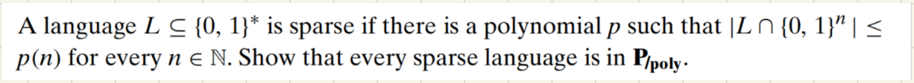
Prove the first part of Theorem 5.4: For every i, if $\mathbf{\Sigma}_i^p = \mathbf{\Pi}_i^p$, then the polynomial hierarchy collapses to the ith level.

[6]



Theorem 5.4

1. *For every* $i \geq 1$, *if* $\mathbf{\Sigma}_i^p = \mathbf{\Pi}_i^p$ *then* $\mathbf{PH} = \mathbf{\Sigma}_i^p$; *that is, the hierarchy collapses to the ith level.*
2. *If* $\mathbf{P} = \mathbf{NP}$ *then* $\mathbf{PH} = \mathbf{P}$; *that is, the hierarchy collapses to* $\mathbf{P}$.

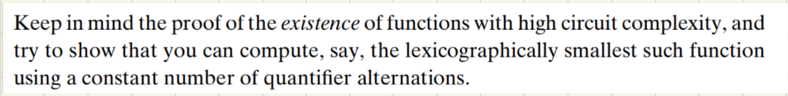
⑧ 

A language $L \subseteq \{0, 1\}^*$ is sparse if there is a polynomial p such that $|L \cap \{0, 1\}^n| \leq p(n)$ for every $n \in \mathbb{N}$. Show that every sparse language is in $\mathbf{P}_{/\mathbf{poly}}$.

[6]

⑨ Use Karp-Lipton theorem to show that for every $k > 0$, $\Sigma_2^p \setminus \mathrm{SIZE}(n^k) \neq \phi$. [9]

Hint:



Keep in mind the proof of the *existence* of functions with high circuit complexity, and try to show that you can compute, say, the lexicographically smallest such function using a constant number of quantifier alternations.
