Mid-Sem Exam

Complexity Theory

25th of February, 2026

Instructions

- You have to answer 4 out of the 5 question sets and can get a maximum of 80 marks.
- You can take a maximum of 2 hours to write this exam.
- The marks you get in this exam will contribute to 20% of your total grade in the course.
- You don't have to reprove any statement from the class notes or any of the problem sets. However, whenever you use any such statement (without proof), you have to mention that it was done in class/was in some problem set. Otherwise, marks will be deducted.

Question Set A

1. Let $a_n = nr^n$ for some $r \in (0,1)$ and $b_n = \frac{1}{n}$. Show that $a_n = o(b_n)$. **[6]**
2. Consider the following two statements about $f : \mathbb{R} \to \mathbb{R}$.
   (a) For every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, $|x| < \delta$ implies $|f(x)| < \epsilon$.
   (b) For every $\epsilon > 0$ and every $x \in \mathbb{R}$, there exists $\delta > 0$ such that $|x| < \delta$ implies $|f(x)| < \epsilon$.

   i. Decide with justification whether the two above statements are equivalent. If not, explore any one-way implications. **[6]**

   ii. Give an example of a function that satisfies (a) and one that doesn't. Redo for (b). **[4 + 4]**

Question Set B

1. (a) Let $\mathcal{C}$ be any circuit computing an n-bit function that contains only AND and OR gates, as well as gates that compute the constant functions 0 and 1.
   Prove that $\mathcal{C}$ must be monotone. That is, show that if $x, x' \in \{0,1\}^n$, with $x_i \leq x'_i$ for every $i \in [n]$, then $\mathcal{C}(x) \leq \mathcal{C}(x')$. **[8]**

   (b) Use part (a) to show that $\{\text{AND}, \text{OR}\}$ is not universal. **[4]**
2. Give an argument for why the following is true: $\mathsf{TIME}(T(n)) \subseteq \mathsf{SIZE}(O((T(n))^2))$. **[4]**

3. Consider the following languages:

$$\mathsf{INDSET}_{0.1n} = \{(V,E) \; : \; (V,E) \text{ has an independent set of size atleast } (0.1\,|V|)\}$$

$$\mathsf{CLIQUE}_{0.1n} = \{(V,E) \; : \; (V,E) \text{ has a clique of size atleast } (0.1\,|V|)\}$$

Show that $\mathsf{INDSET}_{0.1n} \leq_p \mathsf{CLIQUE}_{0.1n}$. **[4]**

Question Set C

1. Recall that NP is the set of all boolean functions $f : \{0,1\}^* \to \{0,1\}$ for which there exist $k \in \mathbb{N}$ and $V \in \mathsf{P}$ such that for every $x \in \{0,1\}^*$,

$$f(x) = 1 \text{ iff } \exists c \in \{0,1\}^{|x|^k} \text{ for which } V(x,c) = 1$$

Also recall that $\mathsf{NTIME}(T(n))$ is the set of all boolean functions $f : \{0,1\}^* \to \{0,1\}$ for which there is a non-deterministic Turing Machine N which has the following properties:

- For every $x \in \{0,1\}^*$, $f(x)$ is the output on some computational branch of N.
- For every $x \in \{0,1\}^*$, every computational branch of N halts in at most $T(|x|)$ many steps.

Show that $\mathsf{NP} = \cup_{k \in \mathbb{N}} \mathsf{NTIME}(n^k)$. **[12]**

2. Recall that coNP is the set of all boolean functions $f : \{0,1\}^* \to \{0,1\}$ for which there exist $k \in \mathbb{N}$ and $V \in \mathsf{P}$ such that for every $x \in \{0,1\}^*$,

$$f(x) = 1 \text{ iff } \forall c \in \{0,1\}^{|x|^k} \text{ for which } V(x,c) = 1$$

Show that $\mathsf{NP} = \mathsf{coNP}$ iff for every $L \in \mathsf{NP}$, the complement of L is in NP. **[3]**

3. Recall that for any language $O \subseteq \{0,1\}^*$, P^O is the set of languages that can be decided by a polynomial-time deterministic TM with oracle access to O.

Let $\mathsf{EXPCOM} = \{(M, x, 1^n) \; : \; M \text{ is a TM that outputs 1 on } x \text{ within } 2^n \text{ steps}\}$. Show that

$$\mathsf{EXP} \subseteq \mathsf{P}^{\mathsf{EXPCOM}}.$$ **[5]**

Question Set D

1. Use the padding argument to prove the following:

$$\mathsf{NTIME}(n^2) \subseteq \mathsf{TIME}(n^3) \implies \mathsf{NTIME}(n^3) \subseteq \mathsf{TIME}(n^{10}).$$ **[8]**

2. (a) Given a 3-CNF, ϕ, design a machine that halts on every input iff ϕ is satisfiable. **[5]**

   (b) Recall the language $\mathsf{HALT}_{\mathsf{ALL}} = \{M \; : \; M \text{ is a TM that halts on every input}\}$. Show that $\mathsf{HALT}_{\mathsf{ALL}}$ is NP-hard. Is it NP-complete? **[5+2]**

Question Set E

Recall that a graph is said to have an *Eulerian Tour* if it is possible to construct a tour (a walk starting and ending on the same vertex) that visits each edge exactly once.

1. Show that if a connected graph has an Eulerian tour then every vertex in it has an even number of incident edges. **[4]**

2. Suppose we are given a connected graph in which every vertex has an even number of incident edges.

   (a) Show that if we start a walk from any vertex, arbitrarily choosing a new edge to take next, then we will always be able to find a cycle. **[4]**

   (b) Use induction (& part (a)) to show that the given graph must have an Eulerian tour. **[4]**

3. Use (1) and (2) to show that the following language is in $\mathsf{NP} \cap \mathsf{coNP}$:

   $$\mathsf{EULERTOUR} = \{(V, E) \ : \ (V, E) \text{ has an Eulerian Tour}\} .$$

   Assume that the universe (for EULERTOUR) is the set of all connected graphs. **[8]**