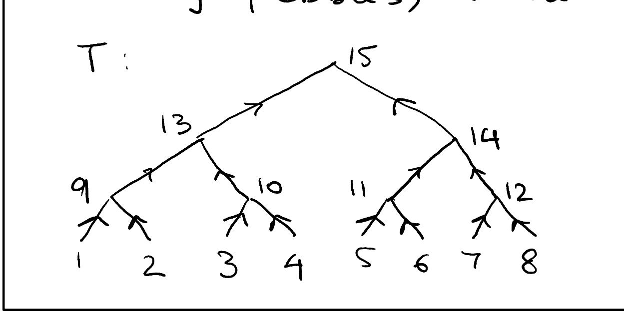
Quiz 9

Let G be rooted DAG. A pebbling game on G is the following:

- Each vertex of G can store atmost one pebble
- The game begins with no pebbles on G.
- In each move, one of the following can be done
  - If all the immediate predecessors of a vertex are pebbled then the vertex can be pebbled.
  - A pebble can be removed from any vertex.
- The game ends when the root is pebbled and all the other vertices are unpebbled.
- The game is said to take h pebbles if h pebbles are required to end the game.

① i) Give an upperbound on the no. of pebbles needed for a game on T (by giving a strategy using that many pebbles) where



ii) Redo (i) for T being the complete binary tree of height h.

[6 + 6]

② Consider the following problem of Circuit Evaluation : that we had encountered earlier

$$\mathrm{EVAL}(\mathcal{C}, x) = \mathcal{C}(x).$$

We had shown that $\mathrm{EVAL} \in \mathsf{P/poly}$.

i) Show that $\mathrm{EVAL} \in \mathsf{P}$.

ii) Relate the space required to compute $\mathrm{EVAL}(\mathcal{C}, x)$ with the no. of pebbles required to play the pebbling game on the underlying graph of $\mathcal{C}$.

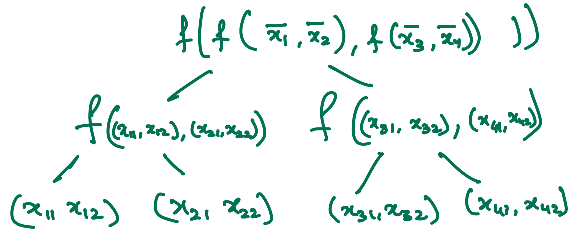
[2 + 6]

Related Recent Developments

Tree Evaluation Problem (TEP) :

Given a function $f : \{0,1\}^{\ell} \times \{0,1\}^{\ell} \to \{0,1\}^{\ell}$, and a complete binary tree of height h, you want to output the function being computed at the root assuming each node is labelled by f and some input is specified at the leaves.

Eg: $f : \{0,1\}^2 \times \{0,1\}^2 \to \{0,1\}^2$



$$\text{Input length} = \underbrace{2^{2\ell}\cdot\ell}_{\text{to specify func.}} + \underbrace{2^{h}\cdot\ell}_{\text{to specify input}} = n$$

$$\Rightarrow h, \ell = O(\log n).$$

Space reqd. in the recursive algo. for TEP $= O(\ell \cdot h)$
$= O(\log^2 n)$

Conjecture : This is tight

Cook - Mertz (2024): TEP $\in$ SPACE$(\log n \cdot \log\log n)$

↳ Son of the Cook from Cook-Levin theorem etc.

fun fact : Sr. Cook was one of the people responsible for the conjecture & there was a \$100 prize amt. attached to its resolution.

Ideas from this UB was used to prove another remarkable inclusion. We have seen that DTIME$(t(n)) \subseteq$ SPACE $(t(n))$. This was improved to DTIME$(t(n)) \subseteq$ SPACE $(t(n)/\log(t(n)))$ in the 70's.

Q: Does there exist $\varepsilon > 0$ s.t.
DTIME$(t(n)) \subseteq$ SPACE $((t(n))^{1-\varepsilon})$.

Ryan Williams : DTIME$(t(n)) \subseteq$
(2025) SPACE $(\sqrt{t(n)\log(t(n))})$

Sipser ('88)
If no (& "some believable statement" is true) then
$P = RP$.
randomness does not give extra power.