Assignment 1

Due: 8:00 am on 18th Aug.

Instructions: Discussion / Websearch / Use of LLM is allowed and infact discussion is encouraged to find solutions.

However the solutions must be written by yourself. Also, all sources used to reach solutions must be mentioned and the solutions you submit must be hand written.

Linear Algebra is a hard pre-requisite for this course. Please revise it if required from your favourite linear algebra book / lecture notes / video series.

① Show that for any $n \in \mathbb{N}$ and any $S \subseteq \mathbb{R}^n$,

$$\mathrm{Span}(S) = \bigcup_{t \geq 0} \left\{ \sum_{i=1}^{t} \alpha_i v_i \; : \; \alpha_i \in \mathbb{R},\, v_i \in S \right\}$$

is a subspace of $\mathbb{R}^n$. [3]

② Let J_n be defined as an $n \times n$ matrix in which every entry is 1.

i) Describe the action this matrix performs in words. That is, for any $v \in \mathbb{R}^n$, how would you describe the vector Jv in words? [1]

ii) What is the kernel of J? [1]

iii) What is the image of J? [1]

(3) i) Show that $L(\mathbb{R}^m, \mathbb{R}^n) = \{ l : \mathbb{R}^m \to \mathbb{R}^n$ is a linear operator $\}$ is a vector space over $\mathbb{R}$ with addition and scalar multiplication defined as follows:

- $(l_1 + l_2)(u) = l_1(u) + l_2(u) \quad \forall\, u \in \mathbb{R}^m$
- $(\alpha l)(u) = \alpha \cdot l(u) \quad \forall\, \alpha \in \mathbb{R},\ u \in \mathbb{R}^m$ [4]

ii) Show that for any $l \in L(\mathbb{R}^m, \mathbb{R}^n)$, there is a matrix M such that

$$Mv = l(v) \quad \forall\ v \in \mathbb{R}^m.$$ [3]

What is the dimension of this matrix? [1]

iii) What is the dimension of $L(\mathbb{R}^m, \mathbb{R}^n)$? Prove your claim. [2]

(4) Show that the rank of an $m \times n$ matrix M is the smallest no. k s.t there exist $m \times k$ & $k \times n$ matrices A, B for which $M = A \times B$. [6]

(5) Show that $v^T A w = \sum_{i,j} A_{ij} v_i w_j$ where

$v \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$, $w \in \mathbb{R}^n$. [2]

(6) Show that $\text{Trace}(AB) = \text{Trace}(BA)$ where $A, B \in \mathbb{R}^{n \times n}$. [2]

(7) Show that for any $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A^T A) = \text{rank}(A)$ [3]

(8) Make sure you know the Gaussian Elimination process for solving a linear system.

i) Why is it easy to solve $Ax = b$ when A is upper triangular? [2]

ii) Suppose you started with a system $Ax = b$ and after you perform the Gaussian elimination process, you got the system $\tilde{A}x = \tilde{b}$. Why is the no. of non-zero rows of $\tilde{A}$ equal to the rank of A? That is, prove why each of the "elementary operations" are rank-preserving. [5]

(9) For any real matrix A, if it is known that $\text{rank}(A)$ ~~$= 0$~~ is not full, under what conditions will the system $A\bar{x} = \bar{b}$ not have solutions? [2]

(10) Suppose $A\bar{x} = \bar{b}$ is a system of linear equations with $A \in \mathbb{R}^{m \times n}$, $\bar{b} \in \mathbb{R}^m$. The system can either have: a) a unique solution

b) infinitely many solutions

c) no solutions.

i) When $m = n$, how do we decide which case (a, b or c) the system satisfies? [2]

ii) Describe the behaviour of the system after the Gaussian elimination process is performed on it in each of the 3 cases. [2]

iii) What is the geometric interpretation of the three cases? [2]

(11) $C \subseteq \mathbb{R}^n$ is said to be convex if $\forall\, x, y \in C$ and $\lambda \in [0,1]$, $\lambda x + (1-\lambda) y \in C$.

i) Show that $\mathcal{C}$ is a set of convex sets then $\bigcap_{C \in \mathcal{C}} C$ is also convex. [3]

ii) Show that $\forall\, \alpha_1, \ldots, \alpha_n \in \mathbb{R}$, $b \in \mathbb{R}$,

$$\{\bar{x} : \sum \alpha_i x_i \leq b\} \subseteq \mathbb{R}^n$$

is a convex set. [2]

iii) Argue why the feasible region in any linear program is always convex. [1]