Mid-Sem Exam

Linear Programming and Combinatorial Optimization

19th of September, 2026

Instructions

- You have to answer all 4 question sets and can get a maximum of 60 marks.
- You can take a maximum of 2 hours to write this exam.
- The marks you get in this exam will contribute to 30% of your total grade in the course.
- You don't have to reprove any statement from the class notes or any of the problem sets. However, whenever you use any such statement (without proof), you have to mention that it was given in the notes/was in some problem set. Otherwise, marks will be deducted.
- If you are caught cheating at any point during the exam, you will be given the minimum possible passing grade in the course irrespective of what marks you score.

Question Set A

1. Two LPs L_1 and L_2 are said to be equivalent iff
   - Any optimal solution of L_1 can be converted into a feasible solution of L_2 with the same objective value.
   - Any optimal solution of L_2 can be converted into a feasible solution of L_1 with the same objective value.

   Convert

   $$\begin{aligned} \max_{x,y} \quad & 30x + 17y \\ \text{s.t.} \quad & 6x + 3y \geq 40 \\ & x - 3y \leq 9 \\ & x \leq 0 \\ & y \in \mathbb{R} \end{aligned}$$

   into the form $\max \mathbf{c}^T\mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} = \mathbf{b};\ \mathbf{x} \geq 0.$

   Show that the converted LP is equivalent to the original one. **[4+6]**

2. Given an $n \times n$ matrix M, consider the following two player game.

   The two players, call them row player and column player, have n strategies each. Row player gets an output M_{ij} if she plays strategy i and column player plays strategy j. We want to find probabilities $p_1, p_2, \cdots, p_n$ for row player which maximizes her output.

   Show that this problem can be formulated as a linear program. **[5]**

Question Set B

1. A Minkowski sum of two sets S_1, S_2 is the set formed by taking all possible sums such that the first vector is from S_1 and the second vector is from S_2.

   $$S_1 + S_2 = \{\mathbf{x} \ : \ \mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 \text{ where } \mathbf{x}_1 \in S_1, \mathbf{x}_2 \in S_2\}.$$

   Prove that $\mathsf{Conv}(S_1 + S_2) = \mathsf{Conv}(S_1) + \mathsf{Conv}(S_2)$ where $\mathsf{Conv}(S)$ denotes the convex hull of the set S, and the $+$s denote Minkowski sums. **[4]**

2. Suppose we are given a graph $G = (V, E, w : V \to \mathbb{R}^+)$ where w is a weight function that assigns a non-negative real number to every vertex. We extend the weight function to subsets of vertices in the most natural way: for any $S \subseteq V$, $w(S) = \sum_{v \in S} w(v)$.

   Let $S_0 \subseteq V$ be such that

   - for every $\{u, v\} \in E$, either $u \in S_0$ or $v \in S_0$
   - $w(S_0) = \min \{w(S) \ : \ S \subseteq V \text{ such that for every } \{u, v\} \in E, \text{ either } u \in S \text{ or } v \in S\}$.

   Give an efficient algorithm (along with a proof of its correctness) to find $S \subseteq V$ such that $w(S) \leq 2 \cdot w(S_0)$ and that for every $\{u, v\} \in E$, either $u \in S$ or $v \in S$. **[5]**

3. Is the following statement true or false?

   If the primal LP is infeasible, then its dual must be unbounded.

   Give either a proof or a counterexample. **[2]**

4. Consider the following LP problem in one dimension:

   $$\min_x \quad c \cdot x \quad \text{subject to} \quad x \geq 0$$

   for some $c > 0$.

   (a) What is the logarithmic barrier function $f_\mu(x)$ for this optimization problem when we use interior point methods? **[1]**

   (b) What is the optimal value of $f_\mu(x)$ for any $\mu > 0$ (Hint: What happens to the derivative of a function at a maxima/minima?)? **[1]**

   (c) Call the optimal value achieved in the previous part as $x^*(\mu)$. What is $\lim_{\mu \to 0^+} x^*(\mu)$? How does it relate to the optimal solution of the original linear program. **[1+1]**

Question Set C

1. Suppose we are given a Simplex tableau as follows:

$$\begin{array}{r|l} \mathbf{x}_B & \mathbf{p} + Q\mathbf{x}_N \\ \hline z & z_0 + \mathbf{r}^T\mathbf{x}_N \end{array}$$

   If $\mathbf{r} \leq \mathbf{0}$, then show that the BFS corresponding to this tableau is an optimal solution of the corresponding LP. **[2]**

2. Consider the following Simplex tableau:

$$\begin{array}{r|lrr} x_3 & 6 & -x_1 & -2x_2 \\ x_4 & 8 & -2x_1 & +x_2 \\ \hline z & & 2x_1 & +3x_2 \end{array}$$

   (a) What are the current basic variables and the corresponding BFS? **[1]**

   (b) Identify the entering variable (use Bland's Rule in case of tie). **[1]**

   (c) Identify the leaving variable (use Bland's Rule in case of tie). **[1]**

   (d) Perform one pivot operation to construct the updated simplex tableau. What is the new objective value? **[2]**

3. Define the dual cone of a cone C to be

$$\widetilde{C} = \left\{ \mathbf{y} \in \mathbb{R}^n \ : \ \mathbf{y}^T\mathbf{x} \geq 0 \text{ for every } \mathbf{x} \in C \right\}.$$

   Show that the dual cone of $\{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{x} \geq \mathbf{0}\}$ is itself. **[3]**

4. A linear program is called self-dual if its dual problem is mathematically identical to the primal problem under a suitable change of variables or symmetric representation.

   Consider the LP (P):

$$\min_{\mathbf{x}} \mathbf{c}^T\mathbf{x} \quad \text{s.t.} \quad M\mathbf{x} \geq -\mathbf{c}, \quad \mathbf{x} \geq \mathbf{0}$$

   where $M \in \mathbb{R}^{n\times n}$ is a skew-symmetric matrix (that is, $M^T = -M$).

   (a) Derive the dual problem (D) of (P) and show that (D) is equivalent to (P). **[2]**

   (b) Show that if (P) has a feasible solution, then its optimal objective value must be 0. **[2]**

   (c) Why can (P) never be unbounded? **[1]**

Question Set D

1. Solve the following problem graphically to find all optimal solutions:

$$\max_{\mathbf{x}} \ 3x_1 + 2x_2 \quad \text{s.t.} \quad x_1 + 2x_2 \leq 6, \quad 2x_1 + x_2 \leq 6, \quad x_1, x_2 \geq 0.$$ **[3]**

2. Given a convex hull $K \subseteq \mathbb{R}^n$ and a point $\mathbf{x}_0 \in \mathbb{R}^n$, a *Separation Oracle* for K is a poly(n) time algorithm that performs one of two actions:

- Asserts Feasibility: Declares that $\mathbf{x}_0 \in K$.
- Returns a Separating Hyperplane: Declares that $\mathbf{x}_0 \notin K$, and outputs a non-zero vector $\mathbf{a} \in \mathbb{R}^n$ and a scalar $b \in \mathbb{R}$ such that

$$\mathbf{a}^T \mathbf{x}_0 > b \quad \text{and} \quad \mathbf{a}^T x \leq b \quad \forall \mathbf{x} \in K$$

Now suppose we are interested in solving

$$\min_{\mathbf{x}} \mathbf{c}^T \mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} \leq \mathbf{b}$$

where A is a $(1.5)^n \times n$ matrix. Further suppose that if $K = \{\mathbf{x} \in \mathbb{R}^n \ : \ A\mathbf{x} \leq \mathbf{b}\}$, then $K = \mathsf{Conv}(S)$ for some $S \subseteq \{0, 1\}^n$.

(a) Can the Simplex algorithm run in time $\text{poly}(n)$? Why or why not? **[2]**

(b) Given access to a Separation Oracle for K, write a pseudocode for an algorithm that solves this LP in time which is $\text{poly}(n, \log(c_{\max}))$. Here $\mathbf{c} = (c_1, \ldots, c_n)$ and $c_{\max} = \max_{i \in [n]} \{|c_i|\}$. **[6]**

(c) If the Separation Oracle has time complexity $T(n)$, then what is the time complexity of the algorithm you gave in part (b)? **[4]**